



ASYMPTOTICAL METHODS FOR HELMHOLTZ OR NAVIER-STOKES TYPE EQUATIONS

Aurélien Klak

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Aurélien Klak

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Méthodes asymptotiques

pour les équations de types

Helmholtz et Navier-Stokes

Thèse soutenue à Rennes le 24/06/2011

devant le jury composé de :

Didier BRESCH

Directeur de recherche université de Savoie/*rapporteur*

Clotilde FERMANIAN KAMMERER

Professeur université de Paris-Est/*rapporteur*

Frédéric ROUSSET

Professeur université de Rennes 1/*examineur*

Xue Ping WANG

Professeur université de Nantes/*examineur*

François CASTELLA

Professeur université de Rennes 1/*co-directeur de thèse*

Christophe CHEVERRY

Professeur université de Rennes 1/*co-directeur de thèse*

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Chapitre I

Introduction

De nombreux phénomènes physiques sont modélisés à l'aide d'un système différentiel dépendant d'un petit paramètre ε . Ce petit paramètre peut représenter une fréquence, une distance, *etc.* qui est petite par rapport aux autres données du problème. Pour simplifier le modèle (diminuer le nombre de variables du système comme dans [3]) ou rendre compte de phénomènes moyens, il est naturel de déterminer un comportement asymptotique de la solution lorsque ε tend vers 0, sa limite par exemple, voire dans les meilleurs des cas justifier un développement asymptotique complet.

Pour étudier ces systèmes, de nombreuses techniques ont été développées suivant le type et la nature des équations (linéaire, non linéaire, *etc.*). On peut par exemple citer le calcul pseudo-différentiel ou l'étude des mesures semi-classiques introduites par Wigner puis développées par P. Gérard [22, 23] ou P.-L. Lions [29] qui ont été utilisées notamment dans l'étude de croisements de modes [17, 18]. Nous pouvons aussi mentionner l'optique géométrique non linéaire. Ce type d'analyse s'est fortement développé ces dernières années notamment pour son succès à étudier des solutions hautes fréquences pour des équations hyperboliques non linéaires, ou décrire certains phénomènes non linéaires comme l'étude des caustiques par exemple.

Dans cette thèse, nous nous intéressons à l'utilisation ou à la mise en œuvre de méthodes pour obtenir des informations asymptotiques sur deux systèmes de nature différente. L'un est linéaire : équation de *Helmholtz* avec source. L'autre est non linéaire : équation de type *Navier-Stokes* forcée par une source polarisée. Dans les deux cas la source se concentre ou oscille fortement lorsque ε tend vers 0. Elle impose un régime particulier aux systèmes.

Pour la première équation, nous étudions le problème de la convergence de la solution de l'équation de *Helmholtz*. On voit notamment que les caractéristiques géométriques des données du système influencent la limite.

Dans un second temps nous étudions la stabilité d'une famille de solutions particulières de l'équation de *Navier-Stokes* notée $\{h_\varepsilon\}_\varepsilon$. Nous produisons un développement asymptotique complet de la solution près de la famille $\{h_\varepsilon\}_\varepsilon$ (lorsque ε tend vers 0) et justifions ensuite la stabilité.

Ces deux problèmes ne sont pas de même nature, et les techniques utilisées et développées pour les traiter sont différentes.

1 Autour d'une équation de Helmholtz

Le chapitre II est dédié à l'étude de l'équation de Helmholtz.

1.1 Contexte

La propagation, en régime harmonique en temps, d'une onde haute fréquence dans un milieu d'indice variable $n^2(x)/\varepsilon^2$ ($\varepsilon > 0$) peut être modélisée à l'aide de l'équation de Helmholtz dite *haute fréquence* :

$$i\varepsilon\alpha_\varepsilon u_\varepsilon(x) + \frac{\varepsilon^2\Delta_x}{2}u_\varepsilon(x) + n^2(x)u_\varepsilon(x) = \frac{1}{\varepsilon^{d/2}}S\left(\frac{x}{\varepsilon}\right), \quad (\text{I.1})$$

où x désigne la variable d'espace parcourant \mathbb{R}^d , et la dimension d est supposée supérieure à 3 dans ce contexte.

- Le coefficient de réfraction du milieu n^2 est supposé régulier. De plus il converge à l'infini vers n_∞^2 : $\lim_{|x| \rightarrow +\infty} n^2(x) = n_\infty^2$. On supposera que $n^2(0)$ est strictement positif.

- Le coefficient d'absorption du milieu α_ε est strictement positif. On suppose de plus qu'il tend vers 0 lorsque ε tend vers 0 : $\lim_{\varepsilon \rightarrow +\infty} \alpha_\varepsilon = 0$.

Le terme $\frac{1}{\varepsilon^{d/2}}S\left(\frac{x}{\varepsilon}\right)$ modélise une source (qui émet un signal) qui se concentre en 0 lorsque $\varepsilon > 0$ tend vers 0. Le même paramètre ε mesure la longueur d'onde typique des ondes qui peuvent se propager dans le milieu d'indice n^2/ε^2 . Un phénomène *d'interactions résonnantes* se produit entre les oscillations hautes fréquences de la source et la vitesse de propagation des ondes imposée par l'indice du milieu.

Ce phénomène étudié initialement par J.-D. Benamou, F. Castella, B. Perthame, T. Katsounis et O. Ruborg [4, 8] a donné lieu à une série de travaux de E. Fouassier dans le cas de deux sources [19], ou dans le cas d'un milieu d'indice discontinu [20]. Citons aussi les travaux de X.P. Wang et P. Zhang [41], qui utilisent une tout autre méthode pour étudier le système (I.1).

Enfin F. Castella [7], pour montrer la convergence de la solution de (I.1), a introduit une approche dite *dépendante du temps*. Elle a donné lieu aux travaux de J.-F. Bony [5] sur l'étude de la mesure de Wigner associée à (I.1) et de J. Royer lorsque le coefficient d'absorption α_ε dépend de la variable d'espace [37]. Nous présentons plus en détail cette approche dans la section 1.2.

Dans [4] l'étude de la mesure de Wigner associée à l'équation de Helmholtz (I.1) s'appuie sur une étude précise de la solution près des lieux où peuvent se produire des *interférences résonnantes constructibles*, ici $x = 0$. Pour ce faire, une remise à l'échelle de la solution de l'équation de Helmholtz est utilisée en posant $w_\varepsilon(x) := \varepsilon^{d/2}u_\varepsilon(\varepsilon x)$. Une démarche analogue est d'ailleurs envisagée dans [8] lorsque la source se concentre près d'une variété Γ (le changement

de variable adéquat est alors de la forme $w_y^\varepsilon := \varepsilon^{d/2} u_\varepsilon(y + \varepsilon x)$ pour $y \in \Gamma$). Cela revient en quelque sorte à mesurer comment se concentre ou oscille la solution u_ε près des points où se concentre la source.

À l'instar de [7] nous étudions par la suite l'équation satisfaite par w_ε :

$$i\varepsilon\alpha_\varepsilon w_\varepsilon(x) + \frac{\Delta_x}{2} w_\varepsilon(x) + n^2(\varepsilon x) w_\varepsilon(x) = S(x), \quad x \in \mathbb{R}^d, \quad d \geq 3. \quad (\text{I.2})$$

L'un des problèmes dans l'étude de (I.2) vient du fait que l'équation

$$\frac{\Delta_x}{2} w_\varepsilon(x) + n^2(\varepsilon x) w_\varepsilon(x) = S(x), \quad x \in \mathbb{R}^d,$$

n'admet pas une unique solution. Le terme $i\varepsilon\alpha_\varepsilon w_\varepsilon(x)$ assure l'existence et l'unicité de la solution de l'équation (I.2) et sélectionne en somme la solution dite sortante de l'équation I.2. Il s'agit en réalité d'une condition de radiation à l'infini qui impose, pour l'essentiel, à la solution w_ε un certain régime oscillant à l'infini.

Lorsque ε tend vers 0, il est facile de montrer que la solution w_ε de (I.2) converge au sens des distributions vers w_{out} solution de :

$$\frac{\Delta_x}{2} w_{out}(x) + n^2(0) w_{out}(x) = S(x), \quad x \in \mathbb{R}^d. \quad (\text{I.3})$$

Pour les mêmes raisons, l'équation (I.3) ne détermine pas w_{out} (problème d'unicité). Cependant le choix particulier du terme $i\varepsilon\alpha_\varepsilon w_\varepsilon(x)$ dans (I.2) laisse à penser que la solution limite est la solution dite *sortante* donnée par

$$\forall x \in \mathbb{R}^d, \quad w_{out}(x) := \lim_{\delta \rightarrow 0^+} \left(i\delta + \frac{\Delta_x}{2} + n^2(0) \right)^{-1} S(x). \quad (\text{I.4})$$

Cette solution est aussi caractérisée comme l'unique solution de l'équation (I.3) vérifiant la condition de radiation de Sommerfeld à l'infini (voir par exemple [14] pour le cas d'un potentiel constant ou [27, 38] dans le cas de potentiel à longue portée ou encore [32]) :

$$\frac{x}{\sqrt{2}|x|} \cdot \nabla_x w_{out}(t) + in(0) w_{out}(x) = O\left(\frac{1}{|x|^2}\right), \quad |x| \longrightarrow +\infty. \quad (\text{I.5})$$

La convergence de w_ε vers w_{out} n'est pas claire a priori. En effet comme $\lim_{|x| \rightarrow +\infty} n^2(\varepsilon x) = n_\infty^2$, la solution en dimension 3 de (I.2) devrait se comporter comme $\exp(i2^{-1/2}n_\infty|x|)/|x|$ pour $|x|$ grand. D'autre part, si on considère que la solution limite suit la condition de Sommerfeld à l'infini, w_{out} devrait asymptotiquement être de l'ordre de $\exp(i2^{-1/2}n(0)|x|)/|x|$ lorsque $|x|$ est grand. En toute généralité, n_∞ est différent de $n(0)$. Ceci est contradictoire et rend les démonstrations de convergence de w_ε vers w_{out} très délicates.

Le premier résultat de convergence de w_ε dans le cas d'un indice non constant (voir [7]) requiert une hypothèse géométrique sur le potentiel n^2 : *l'hypothèse d'absence de refocalisation des rayons* (I.8). Cette hypothèse prévient de trop fortes interactions entre la source oscillante

et l'opérateur $\frac{\Delta_x}{2} + n^2(\varepsilon x)$. Sous cette hypothèse, w_ε converge vers w_{out} solution de l'équation (I.3) vérifiant la condition (I.5).

Le travail proposé ici se place dans la continuité de [7]. Nous nous appuyons très fortement sur les résultats démontrés dans cet article. Nous construisons explicitement un potentiel qui ne vérifie plus la condition de *non-refocalisation des rayons* (II.10). Nous démontrons alors que la solution w_ε converge tout de même, mais vers une perturbation de w_{out} . Cette perturbation caractérise les *interactions résonnantes* et peut être calculée explicitement.

En particulier, la condition d'absence de focalisation des rayons est nécessaire pour obtenir la convergence de w_ε vers w_{out} .

1.2 Résumé des résultats

L'approche introduite par F. Castella dans [7], dite approche dépendante du temps, consiste à voir w_ε solution de (I.2) comme l'intégrale du propagateur classique associé à l'opérateur $i\varepsilon\alpha_\varepsilon + \varepsilon^2/2\Delta_x + n^2(x)$. La solution s'écrit alors :

$$\forall x \in \mathbb{R}^d, \quad w_\varepsilon(x) = i \int_0^{+\infty} e^{-\alpha_\varepsilon t} e^{it(\frac{\Delta_x}{2} + n^2(\varepsilon x))} S(x) dt. \quad (\text{I.6})$$

De même, la solution sortante w_{out} peut être écrite :

$$\forall x \in \mathbb{R}^d, \quad w_{out}(x) := i \int_0^{+\infty} e^{it(\frac{\Delta_x}{2} + n^2(0))} S(x) dt.$$

Ainsi pour montrer le résultat de convergence de w_ε vers w_{out} ([7]) l'idée est de passer à la limite dans l'expression (I.6).

En pratique, pour étudier la convergence de w_ε au sens des distributions on se ramène à l'étude de u_ε par un changement de variable,

$$\forall \phi \in \mathcal{S}(\mathbb{R}^d), \quad \langle w_\varepsilon, \phi \rangle = \langle u_\varepsilon, \varepsilon^{-d/2} \phi(x/\varepsilon) \rangle.$$

Trouver la limite au sens des distributions de w_ε revient alors à déterminer la limite au sens des distributions de u_ε vu à l'échelle semi-classique. De plus, le second membre de (I.1), $\varepsilon^{-d/2} S(x/\varepsilon)$ se concentre à la même échelle en $x = 0$.

Notons $h(x, \xi) = \xi^2/2 - n^2(x)$ le symbole semi-classique associé au hamiltonien $-\varepsilon^2\Delta_x/2 - n^2(x)$. Alors la contribution principale de l'intégrale est portée par le *niveau d'énergie 0* défini par $H_0 := \{(x, \xi) \in \mathbb{R}^{2d}, \text{ tels que } h(x, \xi) = 0\}$.

La difficulté majeure vient de l'étude des *interactions résonnantes* près du point $x = 0$. Du fait de la structure de l'équation, l'énergie transportée sur H_0 le long des trajectoires hamiltoniennes :

$$\begin{cases} \frac{\partial}{\partial t} X(t, x, \xi) = \Xi(t, x, \xi), & X(0, x, \xi) = x, \\ \frac{\partial}{\partial t} \Xi(t, x, \xi) = \nabla_x n^2(X(t, x, \xi)), & \Xi(0, x, \xi) = \xi, \end{cases} \quad \text{avec } (x, \xi) \in H_0, \quad (\text{I.7})$$

peut s'accumuler en 0, et c'est ce phénomène qui peut empêcher la convergence de w_ε vers w_{out} . Dans un premier temps, afin d'éviter ces accumulations d'énergie en grand temps, nous supposons que les courbes hamiltoniennes sur le niveau d'énergie 0 sont non captives (voir définition page 27). Dans un second temps, afin de prévenir une refocalisation des rayons à un même instant $t < +\infty$ (et donc une accumulation d'énergie), une hypothèse de *non-refocalisation des rayons* est introduite (voir [7]).

Définition. On dit qu'un potentiel n^2 satisfait l'hypothèse de refocalisation des rayons si l'ensemble

$$M := \left\{ (t, p, \eta) \in]0, +\infty[\times \mathbb{R}^{2d} \text{ tels que } \frac{\eta^2}{2} = n^2(0), X(t, 0, p) = 0, \Xi(t, 0, p) = \eta \right\} \quad (\text{I.8})$$

est une sous-variété de $\mathbb{R}^{2d} \times]0, +\infty[$ de dimension strictement plus petite que $d - 1$.

Cette hypothèse revient à supposer que les interactions entre le hamiltonien semi-classique de symbole h et les ondes de longueur d'onde ε émises par la source sont faibles. Plus récemment, J.-F. Bony [5] a aussi introduit une hypothèse dite *de viriel* du même type que (I.8) pour montrer la convergence de la mesure de Wigner associée à (I.1) :

$$mes_{d-1} \left\{ \xi \in \sqrt{2n^2(0)} \mathbb{S}^{d-1}; \quad \exists t > 0 \quad X(t, 0, \xi) = 0 \right\} = 0, \quad (\text{I.9})$$

où mes_{d-1} désigne la mesure de surface euclidienne sur la sphère de rayon $\sqrt{2n^2(0)}$ et \mathbb{S}^{d-1} est la sphère de rayon 1. Inspiré par [7], il montre que cette condition est nécessaire pour prouver l'unicité de la mesure limite.

Ici nous construisons un potentiel n^2 non captif défini en (II.12) qui ne vérifie pas la condition de refocalisation. Si la source irradie dans la direction des rayons refocalisants définis en coordonnées hypersphériques par

$$I_{\theta_0} = \left\{ p := (|p|, \theta_1, \dots, \theta_{d-1}), \quad \theta_1 \in]-\theta_0, \theta_0[, \quad \theta_i \in]-\pi, \pi[, \quad \forall i \geq 2, \quad |p| = \sqrt{2n^2(0)} \right\},$$

on montre la convergence de w_ε vers une *perturbation* de w_{out} . On notera $d\lambda$ la mesure euclidienne de surface de I_{θ_0} . Le résultat principal de cette partie est le suivant.

Théorème 1.1. Soit n^2 le potentiel défini au niveau de (II.12) . Alors :

- i) n^2 est un potentiel non captif sur le niveau d'énergie 0, H_0 .
- ii) M est une variété à bord. Sa dimension est $d - 1$.
- iii) Soit une source $S \in \mathcal{S}(\mathbb{R}^d)$ vérifiant l'hypothèse de radiation

$$\text{supp}(\widehat{S}) \subset \{t.\eta \text{ tels que } \eta \in I_{\theta_0}, t > 0\}.$$

Alors,

$$\forall \phi \in \mathcal{S}(\mathbb{R}^d), \quad |\langle w_\varepsilon - w_{out}, \phi \rangle - L_\varepsilon(\phi)| = O(\sqrt{\varepsilon}),$$

Soit la vitesse initiale $p_0 := (\sqrt{2n^2(0)}, 0, \dots, 0)$, on note T_R le temps tel que $X(T_R, 0, p_0) = 0$ (on justifie plus loin l'existence de ce temps de retour), L_ε s'écrit :

$$L_\varepsilon(\phi) = C_{T_R, d} e^{\left(\frac{i}{\varepsilon} \int_0^{T_R} \left(\frac{\Xi(s, 0, p_0)^2}{2} + n^2(X(s, 0, p_0)) \right) ds \right)} \int_{I_{\theta_0}} \widehat{S}(p) \widehat{\phi^*}(-p) d\lambda(p)$$

où $C_{T_R, d}$ est une constante non nulle (définie page 54) et ϕ^* est la conjugaison complexe de l'application ϕ .

Le terme $L_\varepsilon(\phi)$ est le fruit des interférences constructibles. Il est le produit de l'action associée au hamiltonien h avec une distribution tempérée T d'ordre 0 :

$$\forall \phi \in \mathcal{S}(\mathbb{R}^d), \quad \langle T, \phi \rangle = \int_{I_{\theta_0}} \widehat{S}(p) \widehat{\phi^*}(-p) d\lambda(p).$$

Cette distribution traduit la façon dont la source radie vers le potentiel. En particulier si le support de \widehat{S} n'est pas dans le cône $\{t, \eta \mid \text{tels que } \eta \in I_{\theta_0}, t > 0\}$ (donc que la source n'envoie aucun signal vers le potentiel miroir), cette distribution est nulle.

Dans un premier temps (cf. section 2) nous montrons *i)* et *ii)*. L'obtention de *ii)* se fait en calculant explicitement l'ensemble M . L'assertion *iii)* se montre (cf section 3) en étudiant les contributions des différentes échelles de temps de l'intégrale (I.6). La contribution aux temps petits $0 \leq t \leq \varepsilon T_0$ converge vers w_{out} pour T_0 assez grand.

Pour des temps très grands, la contribution de l'intégrale est négligeable. En effet, sous l'hypothèse de non-capture des rayons, le propagateur et la source n'interagissent plus.

Finalement, la contribution en temps moyen $\varepsilon T_0 \leq t \leq T_1$ doit être étudiée avec précision. Un des apports de [7] est l'interprétation de l'intégrale (I.6) comme une intégrale oscillante de la forme de (II.15). On effectue alors une méthode de phase stationnaire. Elle permet de calculer explicitement les interactions résonnantes au travers du terme L_ε . Une des difficultés est de justifier que l'on peut appliquer la méthode de la phase stationnaire.

2 Autour d'une équation de Navier-Stokes

Dans le chapitre III, nous étudions une équation de type Navier-Stokes. Nous nous intéressons aux interactions d'oscillations dans un fluide et analysons les phénomènes macroscopiques qui en découlent.

2.1 Introduction du problème

On considère un problème bi-dimensionnel. Nous supposons que le fluide satisfait l'*approximation des milieux continus*. On imprime au système un régime oscillant en l'agitant à l'aide d'une force fortement oscillante et polarisée.

En notant $t \in \mathbb{R}_+$ la variable de temps et $x := (x_1, x_2) \in \mathbb{R}^2$ la variable d'espace alors $\rho \in \mathbb{R}_+$ la densité et u_1, u_2 les deux composantes de la vitesse du fluide $u := {}^t(u_1, u_2) \in \mathbb{R}^2$

satisfont une équation bi-dimensionnelle de type Navier-Stokes, à l'instar de [6, 28], forcée par une source F_ε :

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla \rho^\gamma = \rho(\mathcal{P}_{\varepsilon, \lambda_\varepsilon} u - F_\varepsilon), \end{cases} \quad u \otimes u := \begin{pmatrix} u_1 u_1 & u_1 u_2 \\ u_1 u_2 & u_2 u_2 \end{pmatrix},$$

où $\varepsilon \in]0, 1]$ est un petit paramètre tendant vers 0. Ici, on modélise la viscosité $\mathcal{P}_{\varepsilon, \lambda_\varepsilon}$ et la force F_ε de la manière suivante :

- La dissipation $\mathcal{P}_{\varepsilon, \lambda_\varepsilon}$ est définie par

$$\mathcal{P}_{\varepsilon, \lambda_\varepsilon} u = {}^t(\mathcal{P}_{\varepsilon, \lambda_\varepsilon}^1 u, \mathcal{P}_{\varepsilon, \lambda_\varepsilon}^2 u) := \mu \varepsilon^2 \Delta_x u + \lambda_\varepsilon \varepsilon^2 \nabla \operatorname{div} u,$$

où $\mu \in \mathbb{R}_+^*$ est fixé alors que $\lambda_\varepsilon \in \mathbb{R}_+^*$ tend vers 0 lorsque ε tend vers 0.

- Étant donnée une fonction C^∞ et périodique h de moyenne nulle

$$h : \mathbb{T} \longrightarrow \mathbb{R}, \quad \mathbb{T} := \mathbb{R}/\mathbb{Z}, \quad h \in C^\infty(\mathbb{T}; \mathbb{R}), \quad \int_{\mathbb{T}} h(\theta) d\theta = 0,$$

on considère la famille oscillante, polarisée suivant la deuxième composante

$$F_\varepsilon(x) = {}^t(0, F_\varepsilon^2)(x) := \varepsilon^{-2} {}^t(0, \mu \partial_{\theta\theta}^2 h(\varepsilon^{-2} x_1)), \quad \varepsilon \in]0, 1].$$

Les oscillations dans la direction x_1 sont ainsi forcées à l'échelle ε^{-2} .

Classiquement (voir par exemple [30]), on introduit les variables d'état $p := \frac{\sqrt{\gamma}}{C} \rho^C$ avec $C := \frac{\gamma-1}{2}$ pour obtenir un système quasi linéaire symétrique :

$$\begin{cases} \partial_t p + u \cdot \nabla p + C p \operatorname{div} u = 0, \\ \partial_t u + u \cdot \nabla u + C p \nabla p = \mathcal{P}_{\varepsilon, \lambda_\varepsilon} u - F_\varepsilon. \end{cases} \quad (\text{I.10})$$

La description des interactions d'oscillations pour le système général (I.10) est difficile. Nous nous intéressons à un problème plus simple. En remarquant que

$$\mathcal{P}_{\varepsilon, \lambda_\varepsilon} {}^t(0, h_\varepsilon) - F_\varepsilon = 0, \quad h_\varepsilon(x) := h(\varepsilon^{-2} x_1), \quad \forall \varepsilon \in]0, 1],$$

la famille oscillante ${}^t(0, 0, h_\varepsilon)$ satisfait le système (I.10) et ce pour tout $\varepsilon \in]0, 1]$. On étudie alors la *stabilité* de la famille de solutions $\{{}^t(0, 0, h_\varepsilon)\}_{\varepsilon \in]0, 1]}$ lorsqu'on la perturbe à l'instant initial ($t = 0$). Plus précisément, on considère des données initiales de la forme :

$$(p, u^1, u^2)(0, x) = (0, 0, h)(\varepsilon^{-2} x_1) + (\varepsilon^\nu q_{0, \varepsilon}, \varepsilon^M v_{0, \varepsilon}^1, \varepsilon^M v_{0, \varepsilon}^2)(\varepsilon^{-2} x_1, \varepsilon^{-1} x_2), \quad (\text{I.11})$$

où $(\nu, M) \in \mathbb{N}^2$ avec ν assez grand et $M \geq 7/2$ (on a aussi $\nu \gg M$), et :

$$(q_{0, \varepsilon}, v_{0, \varepsilon}^1, v_{0, \varepsilon}^2)(\theta, y) \in H^\infty(\mathbb{T} \times \mathbb{R}; \mathbb{R}^3), \quad y := \varepsilon^{-1} x_2 \in \mathbb{R}.$$

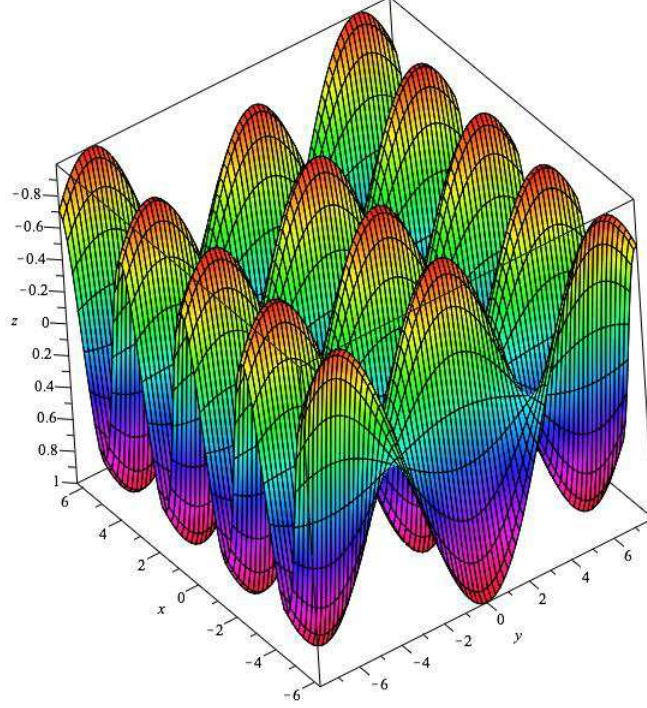


FIG. I.1 – Modèle des oscillations considérées

Un des effets de cette perturbation est l'introduction d'une dépendance en la seconde variable $x_2 \in \mathbb{R}$ (ou $y \in \mathbb{R}$). Bien que ε^ν (ou peut-être aussi ε^M) soit petit, lorsque l'on résout (I.10)-(I.11), on doit comprendre les interactions qui peuvent survenir entre de fortes oscillations dans la direction x_1 (de longueur d'onde ε^2) et des variations rapides dans la direction transversale x_2 (de longueur d'onde ε) (c.f. Figures I.1 et I.2). On est confronté à des problèmes de *turbulence* dans le même esprit que les modèles proposés dans [9, 10, 12].

La structure sous-jacente se précise lorsqu'on réécrit (I.10) en les variables $(\theta, y) \in \mathbb{T} \times \mathbb{R}$. On est confronté à un système hyperbolique-parabolique contenant une partie quasi linéaire symétrique et singulière (en $\varepsilon \in]0, 1]$) :

$$\begin{cases} \partial_t p + \varepsilon^{-2} (u^1 \partial_\theta p + \varepsilon u^2 \partial_y p) + C \varepsilon^{-2} p (\partial_\theta u^1 + \varepsilon \partial_y u^2) = 0, \\ \partial_t u^1 + \varepsilon^{-2} (u^1 \partial_\theta u^1 + \varepsilon u^2 \partial_y u^1) + C \varepsilon^{-2} p \partial_\theta p = \tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon}^1 u, \\ \partial_t u^2 + \varepsilon^{-2} (u^1 \partial_\theta u^2 + \varepsilon u^2 \partial_y u^2) + C \varepsilon^{-1} p \partial_y p = \tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon}^2 u - F_\varepsilon^2, \end{cases} \quad (\text{I.12})$$

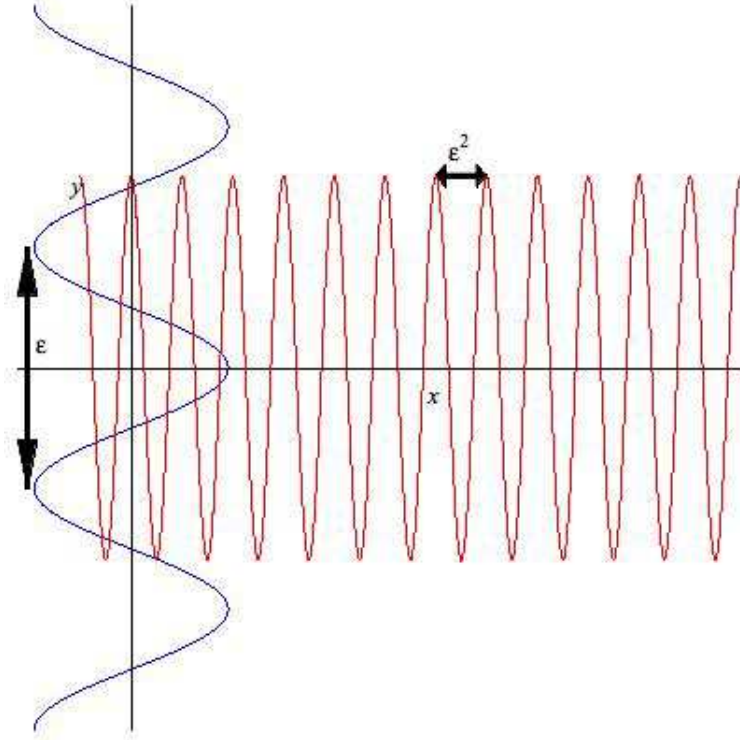


FIG. I.2 – Projection des oscillations considérées suivant l'axe des abscisses et des ordonnées.

et un terme visqueux dégénéré pour la densité $\tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon}$, qui explose lorsque ε tend vers 0 :

$$\tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon} := \begin{pmatrix} \tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon}^1 \\ \tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon}^2 \end{pmatrix} = \frac{1}{\varepsilon^2} \begin{pmatrix} \mu (\partial_{\theta\theta} u^1 + \varepsilon^2 \partial_{yy} u^1) + \lambda_\varepsilon (\partial_{\theta\theta} u^1 + \varepsilon \partial_{\theta y} u^2) \\ \mu (\partial_{\theta\theta} u^2 + \varepsilon^2 \partial_{yy} u^2) + \lambda_\varepsilon (\varepsilon \partial_{\theta y} u^1 + \varepsilon^2 \partial_{yy} u^2) \end{pmatrix}.$$

À ε fixé l'existence d'une solution au problème (I.10)-(I.11) est classique. Montrer que le *problème de Cauchy oscillant* (I.10)-(I.11) est localement bien posé en temps est plus délicat. La difficulté est de gérer les termes singuliers qui peuvent apparaître dans l'équation (I.12).

On montre (proposition III.4) que (pour ν assez grand et $M \geq 7/2$) l'existence d'un temps $T \in \mathbb{R}_+^*$ indépendant de $\varepsilon \in]0, 1]$ tel que $(p_\varepsilon, u_\varepsilon^1, u_\varepsilon^2) = (\varepsilon^\nu q_\varepsilon, \varepsilon^M v_\varepsilon^1, h_\varepsilon + \varepsilon^M v_\varepsilon^2)$ soient solutions de (I.10)-(I.11) sur l'intervalle $[0, T]$. Dans le même temps, on démontre (propositions III.1 et III.2) l'existence d'un développement asymptotique pour $(q_\varepsilon, v_\varepsilon^1, v_\varepsilon^2)$ lorsque ε tend vers 0. Plus précisément, on prouve (dans un sens à préciser) que $q_\varepsilon \simeq q_\varepsilon^a$ et $v_\varepsilon := (v_\varepsilon^1, v_\varepsilon^2) \simeq v_\varepsilon^a := (v_\varepsilon^{a1}, v_\varepsilon^{a2})$ avec

$$q_\varepsilon^a(t, y, \theta) = \sum_{k=0}^{N+1} \varepsilon^k q_k^\varepsilon(t, y, \theta), \quad v_\varepsilon^a(t, y, \theta) = \sum_{k=0}^{N+1} \varepsilon^k \left(v_k^s(t, y, \theta) + v_k^f(\varepsilon^{-2}t, y, \theta) \right). \quad (\text{I.13})$$

Ces développements montrent l'existence d'une couche limite au temps $t = 0$ (du fait du terme $v_k^f(\tau, \cdot)$ cherché décroissant exponentiellement vers 0 lorsque τ tend vers l'infini), conjoint-

tement avec un comportement d'évolution moyen (décrit par v_k^s). Un des aspects à souligner est la production d'une dissipation sur l'équation satisfaite par v_k^s :

$$\partial_t \int_{\mathbb{T}} v_k^s(t, \theta, y) d\theta - \left(\mu + \frac{1}{\mu} \int_{\mathbb{T}} (\partial_\theta^{-1} h)^2(\theta) d\theta \right) \partial_{yy} \int_{\mathbb{T}} v_k^s(t, \theta, y) = S_k(t, y), \quad (\text{I.14})$$

où $\partial_\theta^{-1} h$ est l'unique primitive de h de moyenne nulle et S_k est un terme source qui ne dépend que des v_j^s pour $j \in \{0, \dots, k-1\}$. Cela confirme et justifie que les interactions d'oscillations peuvent être décrites au niveau macroscopique par l'introduction d'une *viscosité turbulente* (comme prévu dans d'autres modèles [31] dits de type $k - \varepsilon$).

2.2 Plan et présentation des méthodes

2.2.1 Construction d'une solution approchée

Dans la première partie du chapitre III, nous construisons une solution approchée du système différentiel (I.10)-(I.11). En écrivant les équations vérifiées par les composantes v_ε^1 et v_ε^2 de la vitesse,

$$\begin{aligned} \mathcal{L}_1(\varepsilon, q_\varepsilon, v_\varepsilon) &:= \partial_t v_\varepsilon^1 + \varepsilon^{-1} h \partial_y v_\varepsilon^1 + \varepsilon^{M-2} (v_\varepsilon^1 \partial_\theta v_\varepsilon^1 + \varepsilon v_\varepsilon^2 \partial_y v_\varepsilon^1) \\ &\quad + C \varepsilon^{2\nu-M-2} q_\varepsilon \partial_\theta q_\varepsilon - \tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon}^1 v_\varepsilon = 0, \end{aligned} \quad (\text{I.15})$$

$$\begin{aligned} \mathcal{L}_2(\varepsilon, q_\varepsilon, v_\varepsilon) &:= \partial_t v_\varepsilon^2 + \varepsilon^{-1} h \partial_y v_\varepsilon^2 + \varepsilon^{-2} \partial_\theta h v_\varepsilon^1 + \varepsilon^{M-2} (v_\varepsilon^1 \partial_\theta v_\varepsilon^2 + \varepsilon v_\varepsilon^2 \partial_y v_\varepsilon^2) \\ &\quad + C \varepsilon^{2\nu-M-1} q_\varepsilon \partial_y q_\varepsilon - \tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon}^2 v_\varepsilon = 0, \end{aligned} \quad (\text{I.16})$$

on remarque que la pression n'intervient qu'au travers du terme $C \varepsilon^{2\nu-M-2} q_\varepsilon^t (\partial_\theta, \varepsilon \partial_y) q_\varepsilon$. Pour $\nu \gg M \geq 7/2$, on peut dans un certain sens (justifié dans la partie III) négliger la pression. En particulier, les variables de pression et de vitesse se *découplent*. Pour construire la vitesse approchée, on étudie plutôt le système (indépendant de la pression) suivant :

$$\begin{cases} \partial_t v_\varepsilon^1 + \varepsilon^{-1} h \partial_y v_\varepsilon^1 + \varepsilon^{M-2} (v_\varepsilon^1 \partial_\theta v_\varepsilon^1 + \varepsilon v_\varepsilon^2 \partial_y v_\varepsilon^1) - \tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon}^1 v_\varepsilon = 0, \\ \partial_t v_\varepsilon^2 + \varepsilon^{-1} h \partial_y v_\varepsilon^2 + \varepsilon^{-2} \partial_\theta h v_\varepsilon^1 + \varepsilon^{M-2} (v_\varepsilon^1 \partial_\theta v_\varepsilon^2 + \varepsilon v_\varepsilon^2 \partial_y v_\varepsilon^2) - \tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon}^2 v_\varepsilon = 0. \end{cases} \quad (\text{I.17})$$

Afin de dégager la hiérarchisation en puissance de ε dans (I.17) on écrit le système comme suit :

$$\partial_t v_\varepsilon + \frac{1}{\varepsilon} \mathcal{T} v_\varepsilon + \frac{1}{\varepsilon^2} \mathcal{Q} v_\varepsilon + \mathcal{L}\mathcal{L}(\varepsilon) v_\varepsilon + \varepsilon^{M-2} \mathcal{N}\mathcal{L}(\varepsilon, v_\varepsilon) = 0 \quad (\text{I.18})$$

avec

$$\begin{aligned} \mathcal{Q} &= \begin{pmatrix} -\mu \partial_{\theta\theta} & 0 \\ \partial_\theta h & -\mu \partial_{\theta\theta} \end{pmatrix}, & \mathcal{T} &:= \begin{pmatrix} h \partial_y - \lambda \partial_{\theta\theta} & 0 \\ 0 & h \partial_y \end{pmatrix}, \\ \mathcal{L}\mathcal{L}(\varepsilon) &:= \begin{pmatrix} -\mu \partial_{yy} & -\lambda \partial_{\theta y} \\ -\lambda \partial_{\theta y} & -(\mu + \varepsilon \lambda) \partial_{yy} \end{pmatrix}, & \mathcal{N}\mathcal{L}(\varepsilon, v_\varepsilon) &:= \begin{pmatrix} v_\varepsilon^1 \partial_\theta v_\varepsilon^1 + \varepsilon v_\varepsilon^2 \partial_y v_\varepsilon^1 \\ v_\varepsilon^1 \partial_\theta v_\varepsilon^2 + \varepsilon v_\varepsilon^2 \partial_y v_\varepsilon^2 \end{pmatrix}. \end{aligned}$$

Le système est non linéaire, vectoriel et singulier en ε . On peut interpréter le système comme une perturbation non autoadjointe de l'opérateur $\mathcal{Q}_0 := -\partial_{\theta\theta} I_2$, où I_2 est la matrice identité en dimension 2. Cette perturbation est de deux types :

- On perturbe \mathcal{Q}_0 à l'ordre 1 par l'opérateur \mathcal{T} . La structure et l'asymptotique de la solution de

$$\partial_t \dot{v}_\varepsilon + \frac{1}{\varepsilon} \mathcal{T} \dot{v}_\varepsilon + \frac{1}{\varepsilon^2} \mathcal{Q}_0 \dot{v}_\varepsilon = 0, \quad (\text{I.19})$$

ont été abondamment étudiées dans le cas scalaire et linéaire. Suivant le choix de \mathcal{Q}_0 (qui peut être différent de notre sélection), on considère alors une équation de transport d'électrons [2], de Focker-Planck [16] ou de Boltzmann [15, 33].

- On perturbe \mathcal{Q}_0 à l'ordre 0 par l'opérateur $\begin{pmatrix} 0 & 0 \\ \partial_\theta h & 0 \end{pmatrix}$. Cette perturbation rend l'opérateur \mathcal{Q} non autoadjoint.

La difficulté ici est triple.

Dans un premier temps, on généralise l'existence d'un développement à deux échelles t et t/ε^2 sous la forme de (I.13) pour le problème vectoriel (I.19). Celui-ci est loin d'être évident même dans le cas scalaire.

Hors du noyau de \mathcal{Q}_0 , le comportement de la solution est dicté par l'opérateur $\varepsilon^{-2} \mathcal{Q}_0$ dont l'échelle d'évolution naturelle semble être t/ε^2 , ce qui justifie la recherche d'un développement sous la forme de (I.13). Dans le noyau, la contribution principale dans (I.18) est alors l'opérateur $\varepsilon^{-1} \mathcal{T}$ qui devrait évoluer à l'échelle intermédiaire t/ε . On explique dans la section (2.1.1) que ce phénomène ne peut se produire. L'hypothèse h de moyenne nulle joue un rôle crucial dans l'élimination des possibles échelles intermédiaires.

Bien que non présentes dans l'ansatz (I.13), les interactions à l'échelle intermédiaire $\varepsilon^{-1}t$ se traduisent par une augmentation de diffusion dans l'équation satisfaite par la limite de v_ε lorsque ε tend vers 0 (*c.f.* équation (I.14)). Les mécanismes mis en jeu dans la création de ce terme diffusif sont du même type que les mécanismes de *dérive-diffusion* (ou *drift-diffusion* en anglais *c.f.* [15, 33]). Dans l'exemple simplifié (I.19) comme les opérateurs \mathcal{T} et \mathcal{Q}_0 sont diagonaux, la discussion devient en réalité scalaire.

Dans un second temps en ajoutant la perturbation non autoajointe d'ordre 0, *i.e.* $\partial_\theta h$, il faut justifier qu'on peut encore obtenir un développement sous la forme de (I.13). La présence de ce terme nécessite une discussion alors vectorielle.

La recherche de profils $v_k^f(\tau, \cdot)$ exponentiellement décroissants en la variable τ ne peut plus être justifiée à l'aide d'une étude spectrale de l'opérateur \mathcal{Q} . Plutôt que de considérer l'opérateur \mathcal{Q} qui n'est ni borné ni auto-adjoint (dans L^2 par exemple), on utilise de façon capitale la structure triangulaire de l'opérateur \mathcal{Q} : on résout la première équation satisfaite par v_ε^1 puis on injecte la solution obtenue dans la seconde pour trouver v_ε^2 . On utilise de façon cruciale les propriétés spectrales de l'opérateur $-\mu \partial_{\theta\theta}$ (auto-adjoint de domaine dense H^2) pour obtenir la décroissance exponentielle.

Un point important de la construction est de justifier que les reports possibles de la première composante de la vitesse dans l'équation sur la deuxième composante, dus à $\partial_\theta h$ dans \mathcal{Q} , ne perturbent pas le mécanisme de production du terme *diffusif*.

La présence du transport dans l'opérateur \mathcal{T} , nécessaire dans la production de diffusion, se fait conjointement avec la perturbation à l'ordre 0 de l'opérateur \mathcal{Q}_0 .

Enfin, on doit aussi examiner les interactions qui peuvent avoir lieu lorsqu'on complète

le problème avec les opérateurs \mathcal{LL} et \mathcal{NL} . D'une part ces opérateurs renforcent le couplage entre les variables v_ε^1 et v_ε^2 . D'autre part, \mathcal{NL} fait apparaître des phénomènes non linéaires qui sont délicats à traiter. Dans un premier temps on suppose que M est assez grand pour s'en affranchir. Puis nous diminuons M afin de faire apparaître ces phénomènes. On doit dans le cas critique $M = 2$ mettre en avant des *phénomènes de transparences* pour conclure.

On étudie simultanément les différents aspects vectoriel, non linéaire et non autoadjoint de \mathcal{Q} , dans la section 2.1, pour justifier la construction d'une vitesse approchée v_ε^a (voir proposition III.1).

Enfin, pour obtenir le développement sur la pression, on injecte v_ε^a , obtenue précédemment, au niveau de l'équation de transport satisfaite par la pression q_ε :

$$\begin{aligned} \mathcal{L}_0(\varepsilon, q_\varepsilon, v_\varepsilon^a) := & \partial_t q_\varepsilon + \varepsilon^{-1} h \partial_y q_\varepsilon + \varepsilon^{M-2} (v_\varepsilon^{1a} \partial_\theta q_\varepsilon + \varepsilon v_\varepsilon^{2a} \partial_y q_\varepsilon) \\ & + C \varepsilon^{M-2} q_\varepsilon (\partial_\theta v_\varepsilon^{1a} + \varepsilon \partial_y v_\varepsilon^{2a}) = 0. \end{aligned} \quad (\text{I.20})$$

Afin de ne pas avoir à discuter les différentes échelles de temps présentes pour la pression $(t, \frac{t}{\varepsilon}, \frac{t}{\varepsilon^2}, \dots)$ et leurs interactions, on résout explicitement le système (I.20) pour obtenir la pression approchée q_ε^a .

Bien entendu un tel procédé ne fournit qu'une solution approchée du système (I.17). On conclut, en contrôlant le terme $C\varepsilon^{2\nu-M-2} q_\varepsilon^t (\partial_\theta, \varepsilon \partial_y) q_\varepsilon$ lorsque ν est grand, que la solution $(q_\varepsilon^a, v_\varepsilon^a)$ ainsi construite approche (en un sens à préciser) aussi le système (I.12).

2.2.2 Problème de stabilité

Dans une seconde partie (Section 3), nous démontrons la stabilité de la solution approchée $(q_\varepsilon^a, v_\varepsilon^a)$. Par la suite on note $(p_\varepsilon^e, u_\varepsilon^e) := (\varepsilon^\nu q_\varepsilon^e, \varepsilon^M v_\varepsilon^{1e}, h + \varepsilon^M v_\varepsilon^{2e})$ la solution exacte du système (I.11)-(I.12).

Soit R un entier naturel. On rappelle que la solution est définie sur l'intervalle $[0, T_\varepsilon]$ où T_ε peut tendre vers 0 lorsque ε tend vers 0. Nous montrons que le temps d'existence T_ε est borné inférieurement par un temps critique T_c . Pour ce faire on pose les nouvelles variables q_ε^R et v_ε^R définies par

$$(q_\varepsilon^e, v_\varepsilon^e) = (q_\varepsilon^a, v_\varepsilon^a) + \varepsilon^R (q_\varepsilon^R, v_\varepsilon^R).$$

Au lieu de travailler sur la variable $(q_\varepsilon^e, v_\varepsilon^e)$ on préfère par la suite travailler sur le reste $(q_\varepsilon^R, v_\varepsilon^R)$. De plus, l'obtention d'un contrôle sur le reste $(q_\varepsilon^R, v_\varepsilon^R)$ justifie la convergence de la solution approchée $(q_\varepsilon^a, v_\varepsilon^a)$ vers la solution exacte jusqu'à l'ordre R .

Le reste $(q_\varepsilon^R, v_\varepsilon^R)$ est la solution du linéarisé de l'opérateur $\mathcal{L} := {}^t(\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2)$ au point $(q_\varepsilon^a, v_\varepsilon^a)$ corrigée d'un terme non linéaire *petit*. On peut aussi traduire le problème en terme des variables initiales p et u . Alors :

$$(p_\varepsilon^e, v_\varepsilon^e) = (0, 0, h) + (\varepsilon^\nu q_\varepsilon^a, \varepsilon^M v_\varepsilon^M) + \varepsilon^R (\varepsilon^\nu q_\varepsilon^R, \varepsilon^M v_\varepsilon^R).$$

La solution $(\varepsilon^\nu q_\varepsilon^R, \varepsilon^M v_\varepsilon^R)$ peut s'interpréter de nouveau comme la solution du linéarisé du système (I.10) au point $(0, 0, h) + (\varepsilon^\nu q_\varepsilon^a, \varepsilon^M v_\varepsilon^M)$ (toujours corrigé d'un terme non-linéaire *petit*). Pour pointer les difficultés et les solutions que nous mettons en œuvre dans la section 3

on simplifie la discussion ici en étudiant la solution du linéarisé de (I.10) au point ${}^t(0, 0, h)$.

Notons $(p_\varepsilon^l, u_\varepsilon^l)$ la solution sur l'intervalle $[0, T_\varepsilon^l]$ du système linéarisé en ${}^t(0, 0, h)$:

$$\begin{cases} \partial_t p_\varepsilon^l + \varepsilon^{-1} h \partial_y p_\varepsilon^l &= S_\varepsilon^{0,l}, \\ \partial_t u_\varepsilon^l + \varepsilon^{-1} h \partial_y u_\varepsilon^l + \boxed{{}^t(0, \varepsilon^{-2} \partial_\theta h u_\varepsilon^{1,l})} - \tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon}(u_\varepsilon^l) &= {}^t(S_\varepsilon^{1,l}, S_\varepsilon^{2,l}), \end{cases} \quad (\text{I.21})$$

où $S_\varepsilon^l := {}^t(S_\varepsilon^{0,l}, S_\varepsilon^{1,l}, S_\varepsilon^{2,l})$ est un terme source dans $L^\infty([0, T_\varepsilon^l], H^\infty(\mathbb{T} \times \mathbb{R}))$ et $(p_\varepsilon^l, u_\varepsilon^l)(0, \cdot) \equiv 0$.

• On essaye une *approche purement hyperbolique* en supposant dans un premier temps que $\mu = \lambda_\varepsilon = 0$. La dissipation $\tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon}$ est alors nulle. Une méthode d'énergie classique sur le système (I.21) aboutit à

$$\forall t \in [0, T_\varepsilon^l], \quad \left\| (p_\varepsilon^l, u_\varepsilon^l)(t, \cdot) \right\|_{L^2(\mathbb{T} \times \mathbb{R})} \lesssim e^{C_\varepsilon t} \sup_{t \in [0, T_\varepsilon^l]} \|S_\varepsilon^l(t, \cdot)\|_{L^2(\mathbb{T} \times \mathbb{R})},$$

où la constante C_ε vérifie :

$$C_\varepsilon \leq C (1 + \varepsilon^{-2} \|\partial_\theta h\|_{L^\infty}). \quad (\text{I.22})$$

La croissance de C_ε en ε^2 est liée au terme singulier encadré dans le système (I.21). Elle indique que la solution peut croître exponentiellement avec t à moins de considérer des temps t de l'ordre de ε^2 . La difficulté, pointée sur l'exemple du linéarisé en ${}^t(0, 0, h)$, de pousser le développement jusqu'à l'ordre 1 se répercute lorsqu'on étudie le système satisfait par $(q_\varepsilon^R, v_\varepsilon^R)$.

• On utilise alors de façon cruciale la structure de la dissipation $\mathcal{P}_{\varepsilon, \lambda_\varepsilon}$ et donc de la parabolicité de l'équation. On casse la structure singulière de l'équation à l'aide d'un changement de variables (singulier). On étudie alors la nouvelle dissipation afin de récupérer un peu de parabolicité (si λ_ε tend assez vite vers 0).

La justification des estimations jusqu'à l'ordre 1 est un équilibre (ou une compétition) entre les aspects hyperboliques singuliers (le terme $\varepsilon^{-2} \partial_\theta h$ encadré dans (I.21)) et la parabolicité de l'équation (la dissipation $\tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon}$). L'aspect parabolique de l'équation a deux conséquences (classiques) pour la vitesse :

- *Phénomène d'absorption* : la viscosité absorbe les termes singuliers en régularité (les termes H^1) mais aussi singuliers par rapport à ε .
- *Phénomène de régularisation* : en plus d'obtenir des estimations $L_t^\infty L_{\theta, y}^2$ on obtient un contrôle de la vitesse dans $L_t^2 H_{\theta, y}^1$.

Le contrôle de la pression est plus problématique. En effet, la pression linéarisée est solution de :

$$q_\varepsilon^l(t, \theta, y) = \int_0^t S_\varepsilon^{0,l}(s, \theta, y - \varepsilon^{-1} h(\theta)(t - s)) ds.$$

La famille $\{q_\varepsilon^l\}_\varepsilon$ n'est pas bornée dans H^1 . La famille $\{\partial_\theta q_\varepsilon^l\}_\varepsilon$ explose comme ε^{-1} dans $L^2(\mathbb{T} \times \mathbb{R})$. On construit alors des espaces anisotropes en ε pour contrôler la famille de solutions.

La différence de structure (due à la viscosité dégénérée suivant la pression) entre la pression et la vitesse nous amène en particulier à étudier séparément ces deux variables : on obtient

dans un premier temps des estimations sur la vitesse qu'on injecte ensuite dans l'équation satisfaite par la pression.

En suivant ce processus, nous montrons d'une part le caractère localement bien posé du système (I.10)- (I.11) et d'autre part que la solution approchée $(q_\varepsilon^a, v_\varepsilon^a)$ est une bonne description asymptotique de la solution exacte $(q_\varepsilon^e, v_\varepsilon^e)$ (*c.f.* proposition III.4).

Chapitre II

Optimalité de l'hypothèse de refocalisation des rayons pour la convergence de la solution de l'équation de Helmholtz haute fréquence

Abstract : We consider the high frequency Helmholtz equation with a given source and a small absorption parameter $\alpha_\varepsilon > 0$. The semi-classical parameter is ε . α_ε goes to 0 as ε goes to 0. We construct a potential n^2 such that the zero-energy level is non trapping for the underlying classical flow. This potential is such that it does not satisfy the *refocusing condition*: the classical trajectory starting from 0 can accumulate energy near the origin. In that case we prove that u_ε , solution of the high frequency Helmholtz equation, rescaled at scale ε near $x = 0$ converges towards a perturbation of the out-going Helmholtz equation (with coefficient frozen in zero). This perturbation is explicitly computed. It is the product of the action associated to the semiclassical hamiltonian with a distribution. This distribution characterizes the contribution of the rays, radiated from the source, which go back to the origin.

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1 Introduction

In this article, we study the convergence as ε approaches 0 of w^ε solution of the following rescaled Helmholtz equation:

$$i\varepsilon\alpha_\varepsilon w_\varepsilon(x) + \frac{\Delta_x}{2}w_\varepsilon(x) + n^2(\varepsilon x)w_\varepsilon(x) = S(x), \quad x \in \mathbb{R}^d, \quad d \geq 3. \quad (\text{II.1})$$

Here α_ε is an absorption parameter, n^2 is an index of refraction and S is a smooth source term. In the sequel, we assume the following:

- α_ε is supposed to be a positive constant which converges to 0 as ε approaches 0:

$$\alpha_\varepsilon \longrightarrow 0^+, \quad \varepsilon \rightarrow 0.$$

- n^2 is supposed to be a non-negative and smooth function: $n^2 \geq 0$, $n^2 \in C^\infty(\mathbb{R}^d)$. Furthermore it satisfies the asymptotic behaviour

$$|\partial_x^\alpha (n^2(x) - n_\infty^2)| \leq C_\alpha \langle x \rangle^{-\rho-\alpha}, \quad \forall \alpha \in \mathbb{N}^d, \quad \forall x \in \mathbb{R}^d, \quad (\text{II.2})$$

with $\rho > 0$ and $n_\infty^2 \geq 0$. In other words, n^2 can be either a short or a long range potential. We assume that $n^2(0)$ is a positive constant.

Our approach is motivated by the study of the solution of the high frequency Helmholtz equation $u_\varepsilon(x) := \varepsilon^{-d/2}w_\varepsilon(x/\varepsilon)$ solution of (II.8).

One problem when dealing with (II.1) is that considering the following equation:

$$\frac{\Delta_x}{2}\dot{w}_\varepsilon(x) + n^2(\varepsilon x)\dot{w}_\varepsilon(x) = S(x), \quad \forall x \in \mathbb{R}^d, \quad (\text{II.3})$$

the solution is not unique. The addition of the term $i\varepsilon\alpha_\varepsilon\dot{w}_\varepsilon$ in Equation (II.3) ensures the uniqueness of the solution and almost selects the out-going solution $\dot{w}_{\varepsilon,out}$ associated to (II.3):

$$\dot{w}_{\varepsilon,out} := \lim_{\delta \rightarrow 0^+} \left(i\delta + \frac{\Delta_x}{2} + n^2(\varepsilon x) \right)^{-1} S(x). \quad (\text{II.4})$$

In the same way letting ε approach 0 in (II.1), the limit solution w_{out} satisfies in a weak-sense (distribution) that

$$\frac{\Delta_x}{2}w_{out} + n^2(0)w_{out} = S. \quad (\text{II.5})$$

Again the limit solution w_{out} is not well determined by Equation (II.5). The particular choice of $i\varepsilon\alpha_\varepsilon\dot{w}_\varepsilon$ may select when ε approaches 0 the corresponding out-going solution to (II.4) defined as

$$\forall x \in \mathbb{R}^d, \quad w_{out}(x) := \lim_{\delta \rightarrow 0^+} \left(i\delta + \frac{\Delta_x}{2} + n^2(0) \right)^{-1} S(x).$$

This solution can also be determined as the solution of (II.5) satisfying the *Sommerfeld radiation* condition at infinity (see [14, 27, 38, 32]),

$$\frac{x}{\sqrt{2}|x|} \cdot \nabla_x w_{out}(t) + in(0)w_{out}(x) = O\left(\frac{1}{|x|^2}\right), \quad |x| \longrightarrow +\infty. \quad (\text{II.6})$$

A priori the convergence of w_ε to w_{out} is not as easy as explained above. First since $\lim_{|x| \rightarrow +\infty} n^2(\varepsilon x) = n_\infty^2$, the solution (in dimension 3) of (II.1) should behave like $\exp(i2^{-1/2}n_\infty|x|)/|x|$ for $|x|$ large enough. Secondly consider w_{out} solution of (II.6). w_{out} is equivalent to $\exp(i2^{-1/2}n(0)|x|)/|x|$ for $|x|$ large. If $n_\infty \neq n(0)$, this is absurd. This is one of the phenomena which make the proof of convergence of w_ε to w_{out} difficult.

In 2005, one result ([7]) confirms the convergence of w_ε to w_{out} . It requires a geometrical assumption on the potential n^2 : *assumption of non-refocalisation of the rays* defined in (II.10). This assumption prevents too strong interaction between the source S and the operator $\Delta_x/2 + n^2(\varepsilon x)$.

The goal of this article is to explicitly construct a potential n^2 which does not satisfy the hypothesis of *non-refocalisation of the rays*. We prove that w_ε still converges. However the limit is a *perturbation* of w_{out} . The perturbation (explicitly computed) traduces resonant phenomena which can appear (we detail it further). Furthermore, it proves the optimality of the assumption introduced in [7] to obtain the convergence to the out-going solution.

1.1 Description - Focusing condition

The study of (II.1) is done by transforming the problem into a time-dependant problem. This approach introduced in [7] has been used since by J.-F.Bony ([5]) to study the Wigner measure associated to (II.8) or by J. Royer ([36]) when the coefficient of absorption α_ε depends on the space variable x . It consists in writing the solution as the integral over the whole time of the propagator associated to $i\varepsilon\alpha_\varepsilon + \Delta_x/2 + n^2(\varepsilon x)$:

$$\forall x \in \mathbb{R}^d, \quad w_\varepsilon(x) = i \int_0^{+\infty} e^{-\alpha_\varepsilon t} e^{it(\frac{\Delta_x}{2} + n^2(\varepsilon x))} S(x) dt. \quad (\text{II.7})$$

In the same way the out-going solution can be written as

$$\forall x \in \mathbb{R}^d, \quad w_{out}(x) := i \int_0^{+\infty} e^{it(\frac{\Delta_x}{2} + n^2(0))} S(x) dt,$$

so the result of the convergence of w_ε to w_{out} consists in passing to the limit in the integral in (II.7). In practice, it turns out that the limit is dictated by $u_\varepsilon(x) := \varepsilon^{-d/2} w_\varepsilon(x/\varepsilon)$ solution of the high frequency Helmholtz equation

$$i\varepsilon\alpha_\varepsilon u^\varepsilon(x) + \frac{\varepsilon^2 \Delta_x}{2} u^\varepsilon(x) + n^2(x) u^\varepsilon(x) = \frac{1}{\varepsilon^{d/2}} S\left(\frac{x}{\varepsilon}\right), \quad \forall x \in \mathbb{R}^d, \quad (\text{II.8})$$

since we have:

$$\forall \phi \in \mathcal{S}(\mathbb{R}^d), \quad \langle w_\varepsilon, \phi \rangle = \langle u_\varepsilon, \varepsilon^{-d/2} \phi(x/\varepsilon) \rangle.$$

In other words, determining the limit of w_ε in the tempered distribution is equivalent to determine the limit of u_ε in the tempered distribution computed at the semiclassical scale ε . Looking at (II.8), the source concentrates or oscillates at the scale $\varepsilon > 0$. The same parameter ε measures the typical wavelength of waves propagating in a medium of index n^2/ε^2 . Resonance interferences appear. As we shall see, the place where this phenomena can occur is the region where the source concentrates. In our case it is the region near the origin $x = 0$.

As standard in semiclassical analysis we define the semiclassical symbol $h(x, \xi) = \frac{\xi^2}{2} - n^2(x)$ (associated to the semiclassical Schrödinger operator $-\frac{\varepsilon^2}{2}\Delta_x - n^2(x)$). The energy concentrates in phase-space (semiclassically) on the zero-energy level $H_0 := \{(x, \xi) \in \mathbb{R}^{2d}, \text{ s.t. } h(x, \xi) = 0\}$. The energy propagates on H_0 along the Hamiltonian trajectories, solutions of the Hamilton equation

$$\begin{cases} \frac{\partial}{\partial t} X(t, x, \xi) = \Xi(t, x, \xi), & X(0, x, \xi) = x, \\ \frac{\partial}{\partial t} \Xi(t, x, \xi) = \nabla_x n^2(X(t, x, \xi)), & \Xi(0, x, \xi) = \xi, \end{cases} \quad \text{with } (x, \xi) \in H_0. \quad (\text{II.9})$$

As mentioned above, the energy transported along the rays can accumulate dangerously near the origin. In [7], two assumptions are done to prevent it.

- Let $x(t)$ be an Hamiltonian trajectory on the zero-energy level. The trajectory can pass an infinite number of times near $x = 0$. Classically, *Hamiltonian trajectories on the zero-energy level are assumed to be non-trapping*. In other words let $(x, \xi) \in H_0$ and $(X(t, x, \xi), \Xi(t, x, \xi))$ be the associated Hamiltonian trajectory. Then,

$$\lim_{|t| \rightarrow +\infty} |X(t, x, \xi)| = +\infty.$$

It prevents one Hamiltonian trajectory to accumulate energy near the origin.

- An other issue is that many Hamiltonian trajectories can refocus (almost) at the same time in 0. To prevent such phenomena a *non-refocusing condition* has to be made. Such an hypothesis has been introduced in [7].

Definition. We say that n^2 satisfies the non-refocusing condition if the set

$$M := \left\{ (t, p, \eta) \in]0, +\infty[\times \mathbb{R}^{2d} \text{ s.t. } \frac{\eta^2}{2} = n^2(0), X(t, 0, p) = 0, \Xi(t, 0, p) = \eta \right\} \quad (\text{II.10})$$

is a submanifold of $\mathbb{R}^{2d} \times]0, +\infty[$ which satisfies $\dim M < d - 1$.

More recently, J.F. Bony ([5]) shows the convergence of the Wigner measure associated to (II.8). He requires a geometrical assumption on the index of refraction in particular a *Viriel condition*:

$$\text{meas}_{n-1} \left\{ \xi \in \sqrt{2n^2(0)}\mathbb{S}^{d-1}; \quad \exists t > 0 \quad X(t, 0, \xi) = 0 \right\} = 0, \quad (\text{II.11})$$

where meas_{n-1} is the euclidian surface measure on $\sqrt{2n^2(0)}\mathbb{S}^{d-1}$ and \mathbb{S}^{d-1} denotes the unit sphere in dimension d .

Furthermore, inspired by [7], he constructs a potential which is non-trapping but does not verify the *Viriel condition*. In this case he proves the non-uniqueness of the limit of the Wigner measure.

Conditions (II.10) and (II.11) just mean that the set of Hamiltonian trajectories on the zero-energy level issued from 0 which go back to 0 is small. Under the *Viriel condition* (II.11) the way to measure the size of this set is in the sense of the measures whereas in the *refocusing condition* (II.10) it leads to write a condition on the dimension of this set. Of course, the *refocusing condition* (II.10) implies the *Viriel condition* (II.11).

We construct a potential in the limiting case, i.e. $\dim M = d - 1$. We can still prove a convergence result in this case.

1.2 Construction of the potential - Main result

The idea behind the refocusing condition is the size of the Hamiltonian trajectories which refocus in 0.

Let us examine the case of the dimension $d = 2$. The idea is the following. Let M_s be a circular mirror centered in 0. Any ray issued from 0 goes back to 0. However such rays are trapped in the mirror. The mirror has to be cut to let the rays escape (Cf figure II.1).

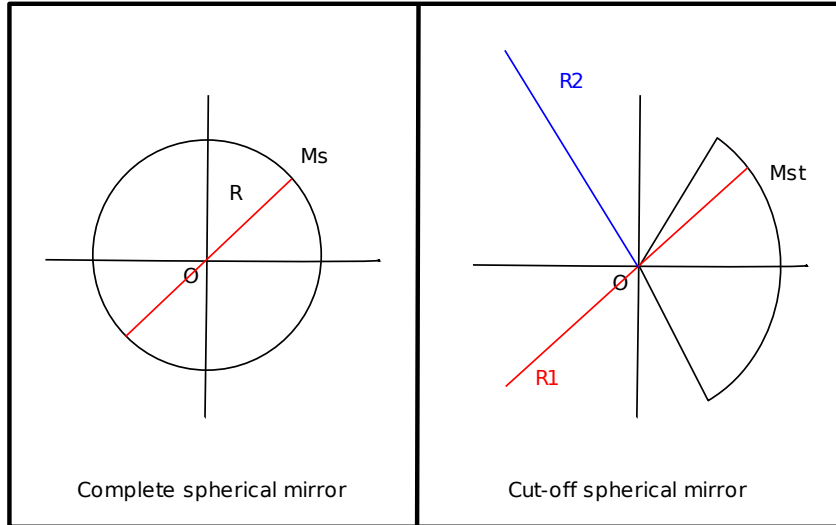


Figure II.1: Spherical mirror in dimension 2

Nevertheless, n^2 has to be relatively smooth (which is not the case for the mirror). We have to

regularize it. We introduce the hyperspherical coordinates $(r, \theta_1, \dots, \theta_{d-1})$ in dimension $d \geq 2$:

$$\begin{aligned} x_1 &= r \cos(\theta_1), \\ x_2 &= r \sin(\theta_1) \cos(\theta_2), \\ x_3 &= r \sin(\theta_1) \sin(\theta_2) \cos(\theta_3), \\ &\vdots \\ x_{d-1} &= r \sin(\theta_1) \dots \sin(\theta_{d-2}) \cos(\theta_{d-1}), \\ x_d &= r \sin(\theta_1) \dots \sin(\theta_{d-2}) \sin(\theta_{d-1}), \end{aligned}$$

with $\theta_1 \in [-\pi, \pi]$, $\theta_j \in [0, 2\pi]$ for $j \neq 1$.

Remark. In dimension $d = 2$, the hyperspherical coordinates are almost the polar coordinates:

$$\begin{cases} x_1 = r \sin(\theta_1), \\ x_2 = r \cos(\theta_1). \end{cases}$$

In dimension $d = 3$, the hyperspherical coordinates are just the spherical coordinates.

First of all, we need to build an approximated spherical mirror. We define a smooth cut-off function χ on \mathbb{R} such that

$$\chi(t) = 1, \quad \forall |t| \leq 1, \quad \chi(t) = 0, \quad \forall |t| \geq 2, \quad \chi(t) \geq 0, \quad \forall t \in \mathbb{R}.$$

Let $R > 0$. We define the spherical mirror of radius R as the function

$$f(x) := \chi(2(r - R)), \quad \forall x = (r, \theta_1, \dots, \theta_{d-1}) \in \mathbb{R}_+ \times [-\pi, \pi] \times [0, 2\pi]^{d-2}.$$

Then we cut it. Let $\theta_0 \in [0, \pi/4]$; we define

$$g(x) := \chi\left(\frac{\theta_1}{\theta_0}\right), \quad \forall x = (r, \theta_1, \dots, \theta_{d-1}) \in \mathbb{R}_+ \times [-\pi, \pi] \times [0, 2\pi]^{d-2}.$$

Finally, let n_∞^2 and μ be two positive constants such that $n_\infty^2 < \mu$ ¹. We set (see figure [II.2](#)):

$$\boxed{n^2(x) := -\mu f(x)g(x) + n_\infty^2}, \quad \forall x \in \mathbb{R}. \quad (\text{II.12})$$

Let (e_1, \dots, e_d) be the canonical basis of \mathbb{R}^d . Let $M_d(\mathbb{R})$ be the space of the square matrices of dimension d . We denote by $\mathbb{O}_d(\mathbb{R})$ the space of orthogonal matrices:

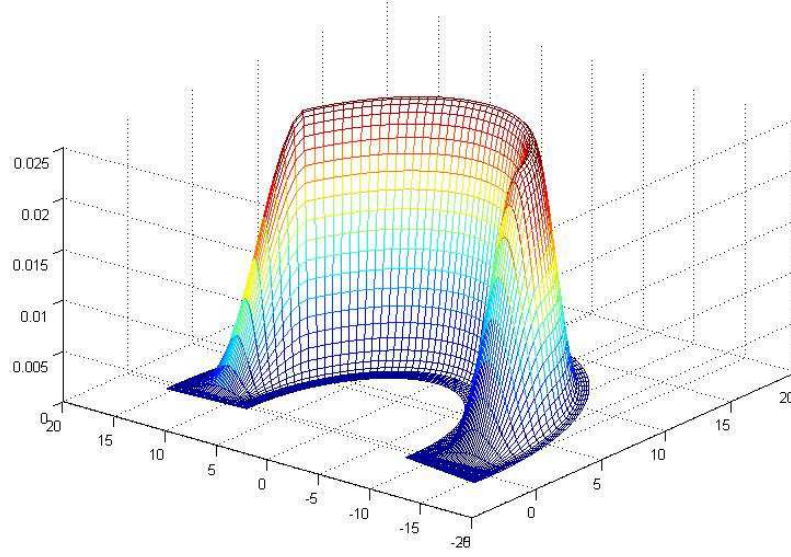
$$\mathbb{O}_d(\mathbb{R}) := \left\{ A \in M_d(\mathbb{R}), \text{ s.t. } \langle Ax, Ay \rangle = \langle x, y \rangle, \forall x, y \in \mathbb{R}^d \right\}.$$

The potential n^2 is invariant under the action of $\mathbb{O}_{d,1}(\mathbb{R})$:

$$\mathbb{O}_{d,1}(\mathbb{R}) := \{ A \in \mathbb{O}_d(\mathbb{R}), \text{ s.t. } Ae_1 = e_1 \}.$$

We introduce a particular set of speeds. It characterizes some specific directions which intuitively correspond to the directions which point at the mirror from the point 0 and which go back to the origin $x = 0$.

¹What can be underlined is that the potential is not non-negative. However, the region of potential where it is negative does not bring difficulties. The study performed in [\[7\]](#) can be done in this context.

Figure II.2: $-n^2(x) + n_\infty^2$ in dimension $d = 2$

Definition. We denote by I_{θ_0} the set of speeds which point towards the potential and go back to the origin:

$$I_{\theta_0} = \{p := (|p|, \theta_1, \dots, \theta_{d-1}), \quad \theta_1 \in]-\theta_0, \theta_0[, \\ \theta_i \in]-\pi, \pi[, \quad \forall i \geq 2, \quad |p| = \sqrt{2n^2(0)}\}. \quad (\text{II.13})$$

In the sequel, we denote by $d\lambda$ the Euclidian surface measure of I_{θ_0} . We can state the main result:

Theorem II.1. Let n^2 be such a potential and $d \geq 2$. Provided that (R, θ_0) satisfies

$$R(1 - \cos(2\theta_0)) - 1/2 < 0, \quad (\text{II.14})$$

then

- i) n^2 is a non-trapping potential on the zero-energy level H_0 .
- ii) M is a submanifold with boundary. Its dimension is $d - 1$.
- iii) We assume that $d \geq 3$. Let $S \in \mathcal{S}(\mathbb{R}^d)$ be a source term satisfying the radiation condition

$$\text{supp}(\widehat{S}) \subset \{t.\eta \quad \text{s.t.} \quad \eta \in I_{\theta_0}, \quad t > 0\}.$$

Then

$$\forall \phi \in \mathcal{S}(\mathbb{R}^d), \quad |\langle w_\varepsilon - w_{out}, \phi \rangle - L_\varepsilon(\phi)| = O(\sqrt{\varepsilon}).$$

Let $p_0 := (\sqrt{2n^2(0)}, 0, \dots, 0)$, we denote by T_R the time such that $X(T_R, 0, p_0) = 0$ (its existence is proved further), L_ε can be written as:

$$L_\varepsilon(\phi) = C_{T_R, d} \varepsilon \left(\frac{i}{\varepsilon} \int_0^{T_R} \left(\frac{\Xi(s, 0, p_0)^2}{2} + n^2(X(s, 0, p_0)) \right) ds \right) \int_{I_{\theta_0}} \widehat{S}(p) \widehat{\phi^*}(-p) d\lambda(p)$$

with $C_{T_R, d}$ a constant different from 0 (defined page 54) and where ϕ^* denotes the complex conjugate of ϕ .

Up to a constant, L_ε is the product of the action associated to $h(x, \xi) = \frac{\xi^2}{2} - n^2(x)$ by T , a distribution of order 0 which registers the rays (send by the source) which go back to 0:

$$\forall \phi \in \mathcal{S}(\mathbb{R}^d), \quad \langle T, \phi \rangle := \int_{I_{\theta_0}} \widehat{S}(p) \widehat{\phi^*}(-p) d\lambda(p).$$

In other words if the source radiates towards the mirror, w_ε converges but no longer to the out-going solution w_{out} . If not, this contribution vanishes. The specific direction (and then the support of S) are chosen so that the rays do not select the boundary of I_{θ_0} , i.e. the direction $|\theta_1| = \theta_0$. This is a technical assumption. It prevents us further from performing a stationary phase theorem on the submanifold with an edge M .

The condition (II.14) is a technical assumption. The result should still be true if we do not assume it.

1.3 Sketch of the proof

The proof of *iii*) in Proposition II.1 follows the steps of the result of convergence [7]. As we use most of the results we recall how it works. For more details, we refer the reader to the corresponding paper.

Let f be a function. In the following for any function f we use the short-hand notation: $f_\varepsilon(x) = \frac{1}{\varepsilon^{d/2}} f\left(\frac{x}{\varepsilon}\right)$. Let ϕ be a function in the Schwartz-space. We have

$$\langle w_\varepsilon, \phi \rangle = \langle u_\varepsilon, \phi_\varepsilon \rangle.$$

In order to use a time dependant approach, u_ε is computed in terms of the semiclassical resolvent $(i\varepsilon\alpha_\varepsilon + \varepsilon^2/2\Delta_x + n^2(x))^{-1}$. It is the integral over the whole time $[0, +\infty[$ of the Schrödinger propagator. It can be written for all $x \in \mathbb{R}^d$,

$$u_\varepsilon(x) = \left(i\varepsilon\alpha_\varepsilon + \frac{\varepsilon^2}{2}\Delta_x + n^2(x) \right)^{-1} S_\varepsilon(x) = i \int_0^{+\infty} \exp \left(it \left(i\varepsilon\alpha_\varepsilon + \frac{\varepsilon^2}{2}\Delta_x + n^2(x) \right) \right) S_\varepsilon(x) dt.$$

Let us define the semiclassical propagator:

$$U_\varepsilon(t) := \exp \left(i \frac{t}{\varepsilon} \left(\frac{\varepsilon^2}{2}\Delta_x + n^2(x) \right) \right) = \exp \left(-i \frac{t}{\varepsilon} H_\varepsilon \right),$$

where H_ε is the semiclassical Schrödinger operator $H_\varepsilon := -\frac{\varepsilon^2}{2}\Delta_x - n^2(x)$. Finally,

$$\langle w_\varepsilon, \phi \rangle = \frac{i}{\varepsilon} \int_0^{+\infty} e^{-\alpha_\varepsilon t} \langle U_\varepsilon(t) S_\varepsilon, \phi_\varepsilon \rangle dt.$$

To pass to the limit, the contribution of the various time scales is analysed in the corresponding integral. They are three main regions:

$$0 \leq t \leq T_0\varepsilon, \quad T_0\varepsilon \leq t \leq T_1, \quad T_1 \leq t \leq +\infty.$$

The contribution on the interval $[T_0\varepsilon, T_1]$ is the most complicated. We represent this contribution as an oscillatory integral with a complex valued phase:

$$\int_{T_0\varepsilon} \int_{\mathbb{R}^{6d}} e^{\frac{i}{\varepsilon}\psi(t,X)} a_\varepsilon(t, X) dt dX.$$

Indeed, if we assume (for instance) that ψ is of the form $\psi(t, \xi) = t\frac{\xi^2}{2}$, the asymptotic is way different for small t and large t . We introduce a last cut-off parameter $\theta \in]T_0\varepsilon, T_1[$.

Finally, the main contribution (at large and moderate times) of the integral comes from the zero-energy level which is excited. Setting the self-adjoint operator $\chi_\delta(H_\varepsilon) := \chi\left(\frac{H_\varepsilon}{\delta}\right)$, the integral on each scale of time and energy can be studied.

1.3.1 Steps of the proof of convergence in [7]

• Contribution at small times •

$$\frac{i}{\varepsilon} \int_0^{2T_0\varepsilon} \chi\left(\frac{t}{T_0\varepsilon}\right) e^{-\alpha_\varepsilon t} \langle U_\varepsilon(t) S_\varepsilon, \phi_\varepsilon \rangle dt.$$

This is the main contribution provided T_0 is large enough. The proof uses the weak convergence of the propagator on S . The result is the following.

Proposition II.2. *Let $n^2(x)$ be bounded and continuous. If S and ϕ belong to $\mathcal{S}(\mathbb{R}^d)$ then:*

1. *For all time $T_0 > 0$,*

$$\frac{i}{\varepsilon} \int_0^{2T_0\varepsilon} \chi\left(\frac{t}{T_0\varepsilon}\right) e^{-\alpha_\varepsilon t} \langle U_\varepsilon(t) S_\varepsilon, \phi_\varepsilon \rangle dt \xrightarrow{\varepsilon \rightarrow 0} i \int_0^{2T_0} \chi\left(\frac{t}{T_0}\right) \left\langle \exp\left(it\left(\frac{\Delta_x}{2} + n^2(0)\right)\right) S, \phi \right\rangle dt.$$

2. *There exists $C_d > 0$ which only depends on the dimension such that:*

$$\left| \frac{i}{\varepsilon} \int_0^{2T_0\varepsilon} \chi\left(\frac{t}{T_0}\right) \langle \exp(it(\Delta_x/2 + n^2(0))) S, \phi \rangle dt - \langle w_{out}, \phi \rangle \right| \leq C_d T_0^{-d/2+1} \xrightarrow{T_0 \rightarrow \infty} 0.$$

• Contribution at large and moderate times away from the zero-energy level •

$$\frac{i}{\varepsilon} \int_{T_0\varepsilon}^{+\infty} e^{-\alpha_\varepsilon t} (1 - \chi)\left(\frac{t}{T_0\varepsilon}\right) \langle (1 - \chi_\delta)(H_\varepsilon) U_\varepsilon(t) S_\varepsilon, \phi_\varepsilon \rangle dt.$$

Away from the zero-energy level, the oscillations of the propagator regularize the contribution. A non-stationary phase argument in time allows to state:

Proposition II.3. *Let n^2 be a long-range potential. Let S and ϕ be in $L^2(\mathbb{R}^d)$. Then there exists a constant $C_\delta > 0$, which only depends on δ , such that for all $\varepsilon \leq 1$ and $T_0 \geq 0$,*

$$\left| \frac{1}{\varepsilon} \int_{T_0\varepsilon}^{+\infty} (1 - \chi)\left(\frac{t}{T_0\varepsilon}\right) \langle (1 - \chi_\delta(H_\varepsilon)) U_\varepsilon(t) S_\varepsilon, \phi_\varepsilon(t) \rangle dt \right| \leq C_\delta \left(\frac{1}{T_0} + \alpha_\varepsilon^2 \right).$$

• Contribution at large times near the zero-energy level •

$$\frac{i}{\varepsilon} \int_{T_1}^{+\infty} e^{-\alpha_\varepsilon t} \langle \chi_\delta(H_\varepsilon) U_\varepsilon(t) S_\varepsilon, \phi_\varepsilon \rangle dt.$$

This contribution is of order $o(\varepsilon^N)$, for all $N \in \mathbb{N}$. The support of $\chi_\delta(H_\varepsilon) U_\varepsilon(t) S_\varepsilon$ and ϕ_ε are examined. For T_1 large enough, the supports of the two functions are disconnected. ϕ_ε is supported near 0 whereas using an Egorov's theorem in large time $\chi_\delta(H_\varepsilon) U_\varepsilon(t) S_\varepsilon$ is microlocally supported near the bicharacteristics. The non-trapping behaviour of n^2 allows to conclude.

Proposition II.4. *Let n^2 be a long-range potential which is non-trapping and satisfies (II.2). Let S and ϕ be in $\mathcal{S}(\mathbb{R}^d)$. Then there exist $\delta_0 > 0$ and $T_1(\delta_0) > 0$ such that for all time $T_1 \geq T_1(\delta_0)$, there exists a constant C_δ such that asymptotically,*

$$\left| \frac{1}{\varepsilon} \int_{T_1}^{+\infty} (1 - \chi) \left(\frac{t}{T_1} \right) e^{-\alpha_\varepsilon t} \langle \chi_\delta(H_\varepsilon) U_\varepsilon(t) S_\varepsilon, \phi_\varepsilon \rangle dt \right| \leq C_\delta \varepsilon.$$

• Contribution at moderate times near the zero-energy level •

$$\frac{i}{\varepsilon} \int_{T_0\varepsilon}^{T_1} e^{-\alpha_\varepsilon t} (1 - \chi) \left(\frac{t}{T_0\varepsilon} \right) \langle U_\varepsilon(t) \chi_\delta(H_\varepsilon) S_\varepsilon, \phi_\varepsilon \rangle dt.$$

The main difficulty comes from this term. Unlike the previous case, the support of $U_\varepsilon(t) \chi_\delta(H_\varepsilon) S_\varepsilon$ and ϕ_ε may intersect. A dangerous accumulation of energy could appear. We summarize this step; yet since it is the key point of our study we give more details in Section 3.

A wave packet approach introduced by Combescure and Robert ([13]) is performed. This allows to compute explicitly $U_\varepsilon S_\varepsilon$ in terms of the linearized flow of $\xi^2/2 - n^2(x)$. Thus, an integral representation with a complex valued phase function is obtained:

$$\frac{1}{\varepsilon^{(5d+2)/2}} \int_{T_0\varepsilon}^{T_1} \int_{\mathbb{R}^{6d}} e^{\frac{i}{\varepsilon} \psi(t, X)} a_N(t, X) dt DX, \quad (\text{II.15})$$

with $X = (q, p, x, y, \xi, \eta)$. The amplitude a_N is defined page 50 and ψ the complex phase is explicitly computed page 50.

As mentionned above, we decompose this integral in two pieces selecting a constant θ in the interval $[\varepsilon T_0, T_1]$. On $[T_0\varepsilon, \theta]$, a change of variables and a stationary phase argument (with semiclassical parameters $\frac{t}{\varepsilon}$) allow to prove the following proposition.

Proposition II.5. *Let n^2 be long-range potential which is non-trapping. For θ small enough, there exists $C_\theta > 0$ such that for all $\varepsilon \leq 1$ we have*

$$\frac{1}{\varepsilon} \int_{T_0\varepsilon}^{\theta} \chi \left(\frac{t}{\theta} \right) \left(1 - \chi \left(\frac{t}{T_0\varepsilon} \right) \right) e^{-\alpha_\varepsilon t} \langle U_\varepsilon(t) \chi_\delta(H_\varepsilon) S_\varepsilon, \phi_\varepsilon \rangle dt \leq C_\theta T_0^{-d/2+1} \xrightarrow{T_0 \rightarrow \infty} 0. \quad (\text{II.16})$$

1.3.2 Contents

The study on $[\theta, T_1]$ is similar to [7]. Yet the contribution is no longer vanishing. We use a stationary phase approach for the oscillatory integral. We prove that the stationary set associated with ψ (defined as $\{(t, X), \nabla_{t,X}\psi(t, X) = 0\}$) is the following

Definition.

$$M_X := \left\{ (t, X) \in]0, +\infty[\times \mathbb{R}^{6d} \text{ s.t. } x = y = q = 0, \xi = p \right\} \\ \bigcap \left\{ (t, X) \in]0, +\infty[\times \mathbb{R}^{6d} \text{ s.t. } (t, p, \eta) \in M \right\}. \quad (\text{II.17})$$

If this set is a submanifold, we can apply the stationary phase theorem to M_X . Nevertheless a condition of non-degeneracy is required for ψ . We denote by (H_2) the *transversality condition* (II.18):

$$\text{Ker } D^2\psi|_m = T_m M_X, \quad \forall m \in \overset{\circ}{M}_X. \quad (\text{II.18})$$

The condition on the support of \widehat{S} is such that the boundary of M_X does not interfere in the oscillatory integral. This is purely technical. The stationary phase theorem with an edge required some non-degeneracy of $D^k\psi$ for some $k \geq 3$ that we have not proved.

Thus, Section 2 is dedicated to the proof of properties on the potential n^2 .

First of all we check that the potential is non-trapping on the zero-energy level (see Subsection 2.1).

Then in Subsection 2.2, we compute the stationary set M_X . In particular we prove that it is a submanifold of $]0, +\infty[\times \mathbb{R}^{6d}$ of dimension $d - 1$ with boundary (part *ii*) in Proposition II.1). Finally we prove that the *transversality condition* is satisfied.

In Section 3 we recall how to obtain the oscillatory integral II.15 (Subsection 3.2). Then we compute the contribution of (II.15) on $[\theta, T_1]$ applying a stationary phase argument (Subsection 3.3).

2 Properties of the potential

2.1 Non-trapping behaviour

The goal of this subsection is to prove that the potential $n^2(x) = -\mu f(r)g(\theta_1) + n_\infty^2$ is *non-trapping* on the zero-energy level H_0 where

$$H_0 := \left\{ (x, \xi) \in \mathbb{R}^{2d}, \text{ s.t. } \frac{\xi^2}{2} = n^2(x) \right\}, \\ = \left\{ (x, \xi) \in \mathbb{R}^{2d}, \text{ s.t. } x = (r, \theta_1, \dots, \theta_{d-1}), \frac{\xi^2}{2} = n_\infty^2 - \mu f(r)g(\theta_1) \right\}.$$

Definition. We denote by B_\emptyset , the gap of potential, the set

$$B_\emptyset := \left\{ x \in \mathbb{R}^d, \text{ s.t. } n^2(x) < 0 \right\} = \left\{ x = (r, \theta_1, \dots, \theta_{d-1}), \text{ s.t. } n_\infty^2 < \mu f(r)g(\theta_1) \right\}.$$

Denoting by $\Pi_x H_0$ the projection of H_0 on the space variable x . We clearly have

$$\Pi_x H_0 = \mathbb{R}^d \setminus B_\emptyset.$$

In other words this set is never reached by any trajectory on the zero-energy level.

The study of the Hamiltonian trajectory is interesting in the region of space where the potential is not constant (see Figure II.4):

Definition. We defined the set, called bump of potential,

$$\begin{aligned} B_p &:= \left\{ x \in \mathbb{R}^d, \text{ s.t. } n^2(x) - n_\infty^2 \leq 0 \right\}, \\ &= \left\{ x := (r, \theta_1, \dots, \theta_{d-1}), \text{ s.t. } R-1 \leq r \leq R+1, -2\theta_0 \leq \theta_1 \leq 2\theta_0 \right\}. \end{aligned}$$

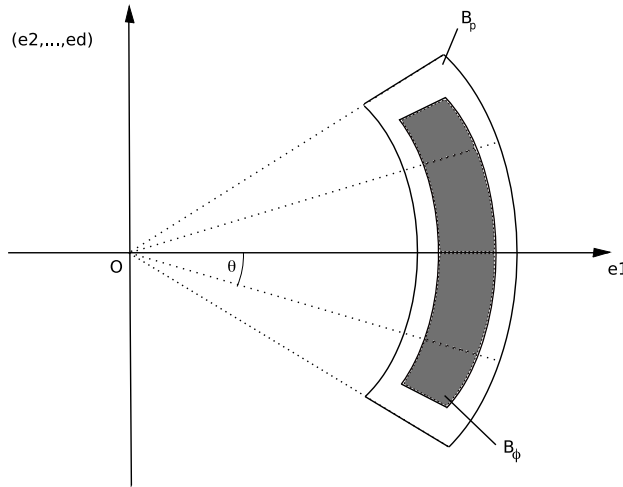


Figure II.3: Bump of potential and gap of potential

Indeed out of B_p the trajectories are just straight-lines. In the sequel, we denote the usual Euclidean scalar product and the usual Euclidean norm by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ respectively. We prove that:

Lemma II.6. Assume condition (II.14) is satisfied. Let $x(t)$ be an Hamiltonian trajectory on the zero-energy level and $x_0 := (R, \dots, 0)$. There exist α a positive constant, β and γ real numbers such that

$$\forall t \in \mathbb{R}, \quad \|x(t) - x_0\|_2^2 \geq \alpha t^2 + \beta t + \gamma.$$

Proof of Lemma II.6.

• **Step one** • We compute the second derivative of $\|x(t) - x_0\|_2^2$ and get

$$\begin{aligned} \frac{1}{2} \frac{d^2}{dt^2} \|x(t) - x_0\|_2^2 &= \left\langle \frac{d^2}{dt^2} x(t), x(t) - x_0 \right\rangle + \left\| \frac{dx}{dt}(t) \right\|^2, \\ &= \langle \nabla n^2(x(t)), x(t) - x_0 \rangle + \left\| \frac{dx}{dt}(t) \right\|^2, \\ &= \langle \nabla n^2(x(t)), x(t) - x_0 \rangle + n^2(x(t)). \end{aligned} \quad (\text{II.19})$$

Since the Hamiltonian trajectory x belongs to H_0 , it justifies the last equality. We develop the first term. Let $x = r \vec{u}_r$ and $x_0 = (R, 0, \dots, 0)$ in Cartesian coordinates. Then

$$\begin{aligned} \langle \nabla n^2(x), x - x_0 \rangle &= \left\langle -\mu f'(r) g(\theta_1) \vec{u}_r - \mu \frac{f(r)}{r} g'(\theta_1) \vec{u}_{\theta_1}, r \vec{u}_r - R \vec{e}_1 \right\rangle, \\ &= F_r(r, \theta_1) + F_\theta(r, \theta_1), \end{aligned}$$

with

$$F_r(r, \theta_1) = -\mu f'(r) g(\theta_1) (r - R \cos(\theta_1)), \quad F_\theta(r, \theta_1) = -\mu \frac{R}{r} f(r) \sin(\theta_1) g'(\theta_1),$$

so that:

$$\frac{1}{2} \frac{d^2}{dt^2} \|x(t) - x_0\|_2^2 = F_r(r, \theta_1) + F_\theta(r, \theta_1) + n^2(x(t)).$$

We notice that n^2 and F_θ are clearly non-negative function on $\Pi_x H_0$. There remains to estimate F_r .

• **Step two: non-negativity of F_r** • First, on the complementary of B_p , F_r is zero so it is non-negative. In the same way on the set $\{R - 1/2 \leq r \leq R + 1/2\}$, $f' \equiv 0$. Thus, $F_r \equiv 0 \geq 0$ on $\{R - 1/2 \leq r \leq R + 1/2\}$.

There remains to study two cases on B_p . We start with the non-negativity of F_r on the set $\{(r, \theta_1, \dots, \theta_{d-1}), \text{ s.t. } R - 1 \leq r \leq R - 1/2, -2\theta_0 \leq \theta_1 \leq 2\theta_0\}$. We have

$$r - R \cos(\theta_1) \leq R - \frac{1}{2} - R \cos(\theta_1) \leq R(1 - \cos(2\theta_0)) - \frac{1}{2} \leq 0,$$

Furthermore, since $f' \geq 0$ on $\{(r, \theta_1, \dots, \theta_{d-1}), \text{ s.t. } R - 1 \leq r \leq R - 1/2\}$, we get

$$F_r \geq 0, \quad \text{on } \{(r, \theta_1, \dots, \theta_{d-1}), \text{ s.t. } R - 1 \leq r \leq R - 1/2, -2\theta_0 \leq \theta_1 \leq 2\theta_0\}.$$

A similar computation proves that

$$F_r \geq 0, \quad \text{on } \{(r, \theta_1, \dots, \theta_{d-1}), \text{ s.t. } R + 1/2 \leq r \leq R + 1, -2\theta_0 \leq \theta_1 \leq 2\theta_0\}.$$

And so F_r is a non-negative function.

• **Step three: Decomposition** • We have just proved that (II.19) is non-negative, nevertheless we wish to obtain a lower (positive) bound. The potential n^2 is non-negative. It is zero on the boundary of B_\emptyset :

$$\partial B_\emptyset := \{(r, \theta_1, \dots, \theta_{d-1}), \quad f(r)g(\theta_1) = n_\infty^2/\mu\}.$$

In the same time, the gradient of the potential is zero near the complementary of B_p . It is positive near ∂B_\emptyset . That is why we introduce the set (a piece of ring)

$$C_{\alpha,\beta} := \left\{ x = (r, \theta_1, \dots, \theta_{d-1}) \in [0, +\infty[\times]-\pi, \pi[\times [0, 2\pi[^{d-2}, \right. \\ \left. \text{s.t. } R - \alpha \leq r \leq R + \alpha, \quad -\beta \leq \theta_1 \leq \beta \right\}.$$

There exist $r_p \in]0, 1[$ and $\theta_p \in]\theta_0, 2\theta_0[$ such that: $B_\emptyset \subset C_{r_p, \theta_p} \subset B_p$.

By construction, there exists c_{n^2} a positive constant such that: $\forall x \in C_{r_p, \theta_p}^c, \quad n^2(x) > c_{n^2} > 0$.

• **Step four: minoration on C_{r_p, θ_p}** • Now we minorate $\langle \nabla n^2(x), x - x_0 \rangle$ on the set C_{r_p, θ_p} . To this end, we introduce a set of the form $C_{\alpha, \beta}$ which lies in B_\emptyset . Let $\eta \in]0, 1[$ such that $(1 - \eta)^2 \in]\frac{n_\infty^2}{\mu}, 1[$. There exists a unique $\delta \in]0, 1[$ such that

$$\chi(1 + \delta) = (1 - \eta).$$

With this choice of δ , the set $C_{\frac{\delta+1}{2}, \theta_0(1+\delta)}$ lies in B_\emptyset . Indeed let $(r, \theta_1, \dots, \theta_{d-1}) \in C_{\frac{\delta+1}{2}, \theta_0(1+\delta)}$:

$$n_\infty^2 - \mu f(r)g(\theta_1) \leq n_\infty^2 - \mu f\left(R + \frac{1+\delta}{2}\right)g(\theta_0(1+\delta)), \\ \leq n_\infty^2 - \mu \chi(1+\delta)\chi(1+\delta) \leq n_\infty^2 - \mu(1-\eta)^2 < 0.$$

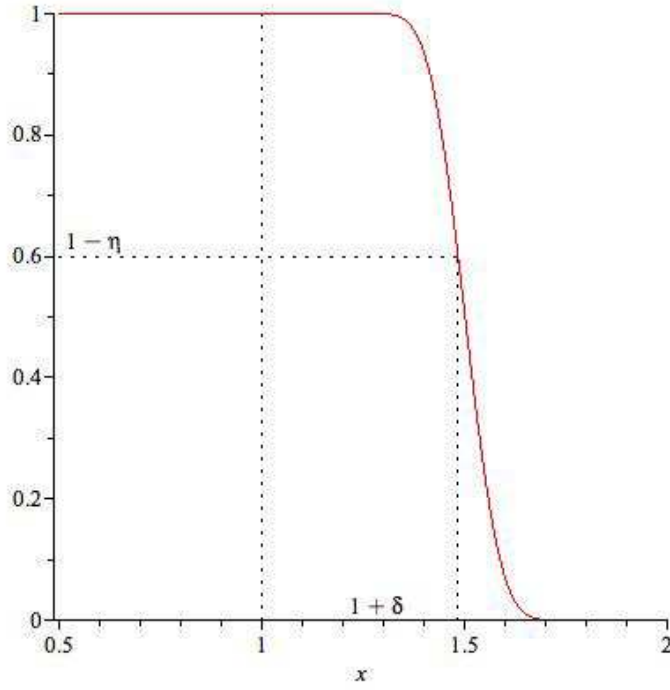
It is sufficient to study the problem on the set $C_{r_p, \theta_p} \setminus C_{\frac{\delta+1}{2}, \theta_0(1+\delta)}$. We decompose the study on C_{r_p, θ_p} depending on the action of the potential (see Figure II.5): $C_{r_p, \theta_p} = Z_r^1 \cup Z_r^2 \cup Z_\theta^1 \cup Z_\theta^2$ with

$$Z_r^1 := \left\{ x \in \mathbb{R}^d, \text{ s.t. } R - r_p \leq r < R - (\delta + 1)/2, \quad -\theta_0(1 + \delta) \leq \theta_1 \leq \theta_0(1 + \delta) \right\} \cap B_\emptyset^c, \\ Z_r^2 := \left\{ x \in \mathbb{R}^d, \text{ s.t. } R + (1 + \delta)/2 \leq r < R + r_p, \quad -\theta_0(1 + \delta) \leq \theta_1 \leq \theta_0(1 + \delta) \right\} \cap B_\emptyset^c,$$

and

$$Z_\theta^1 := \left\{ x \in \mathbb{R}^d, \text{ s.t. } R - r_p \leq r < R + r_p, \quad -2\theta_p \leq \theta_1 \leq -\theta_0(1 + \delta) \right\} \cap B_\emptyset^c, \\ Z_\theta^2 := \left\{ x \in \mathbb{R}^d, \text{ s.t. } R - r_p \leq r < R + r_p, \quad \theta_0(1 + \delta) \leq \theta_1 \leq 2\theta_p \right\} \cap B_\emptyset^c.$$

On Z_r^1 . We use the structural hypothesis (II.14) to get

Figure II.4: Choice of δ

$$\begin{aligned}
 F_r &= -\mu f'(r)g(\theta_1)(r - R \cos(\theta_1)) \\
 &\geq -\mu f'(r)g(\theta_1)\left(R - \frac{1+\delta}{2} - R \cos(2\theta_0)\right) \geq \frac{\mu\delta}{2} f'(r)g(\theta_1). \quad (\text{II.20})
 \end{aligned}$$

On $[R - r_p, R - (\delta + 1)/2]$, f is an increasing function. Since $[R - r_p, R - (\delta + 1)/2]$ is a compact set, there exists c_{f,Z_r^1} a positive constant such that

$$\forall r \in [R - r_p, R - (\delta + 1)/2], \quad f'(r) \geq c_{f,Z_r^1}.$$

By construction of the function χ we get that there exists c_{g,Z_r^1} such that

$$\forall \theta_1 \in]-\theta_0(1+\delta), \theta_0(1+\delta)[, \quad g(\theta_1) \geq c_{g,Z_r^1}.$$

Finally putting those estimates in minoration (II.20), we obtain

$$\forall (r, \theta_1, \dots, \theta_{d-1}) \in Z_r^1, \quad F_r \geq \frac{\mu\delta}{2} c_{f,Z_r^1} c_{g,Z_r^1} := c_{Z_r^1}.$$

On Z_r^2 the proof is the same. There exists a positive constant $c_{Z_r^2}$ such that

$$\forall (r, \theta_1, \dots, \theta_{d-1}) \in Z_r^2, \quad F_r \geq c_{Z_r^2}.$$

On Z_θ^1 . We now expect F_θ to dictate the behaviour. First on $[R - r_p, R + r_p]$, f is a positive function. There exists a positive constant c_{f,Z_θ^1} such that

$$\forall r \in [R - r_p, R + r_p], \quad f(r) \geq c_{f,Z_\theta^1}.$$

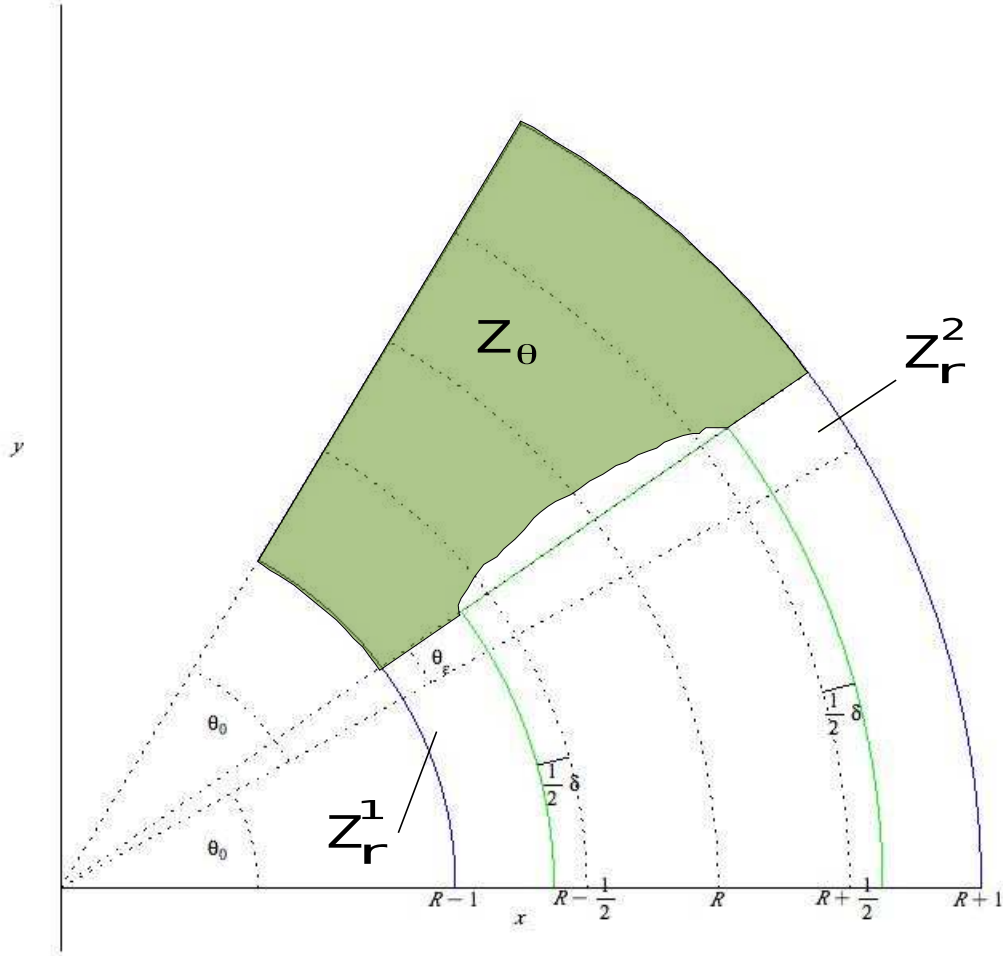


Figure II.5: Zone of study

On $[-2\theta_p, -\theta_0(1+\delta)]$, the function g is increasing. There exists a positive constant c_{g,z_θ^1} such that

$$\forall \theta_1 \in [-2\theta_p, -\theta_0(1+\delta)], \quad g'(\theta_1) \geq c_{g,z_\theta^1}.$$

Finally, we get

$$\forall (r, \theta_1, \dots, \theta_{d-1}) \in Z_\theta^1, \quad F_\theta \geq -\mu \frac{R}{r} f(r) g'(\theta_1) \sin(\theta_1) \geq \mu \frac{R}{R+1} c_{f,z_\theta^1} c_{g,z_\theta^1} \sin(\theta_0) = c_{Z_\theta^1}.$$

In the same spirit, there exists a positive constant $c_{Z_\theta^2}$ such that:

$$\forall (r, \theta_1, \dots, \theta_{d-1}) \in Z_\theta^2, \quad F_\theta \geq c_{Z_\theta^2}.$$

Conclusion : There exists a positive constant $C_\nabla := \min(C_{Z_r^1}, C_{Z_r^2}, C_{Z_\theta^1}, C_{Z_\theta^2})$ such that:

$$\forall x \in B_{r_p, \theta_p} \setminus C_{\frac{\delta+1}{2}, \theta_0(1+\delta)}, \quad \langle \nabla n^2(x), x - x_0 \rangle \geq C_\nabla.$$

• **Step five: end of the proof** • Putting all the information together, we obtain:

$$\forall x \in \Pi_x H_0, \quad \langle \nabla n^2(x), x - x_0 \rangle + n^2(x) \geq \min(c_{n^2}, c_\nabla) =: \alpha > 0.$$

Thus,

$$\forall t \in \mathbb{R}, \quad \frac{d^2}{dt^2} \|x(t)\|^2 \geq \alpha.$$

Integrating twice proves the result. ■

Corollary II.7. *Assuming condition (II.14) is satisfied, then n^2 is a non-trapping potential on the zero-energy level.*

Proof of corollary II.7 We apply the preceding lemma and we let t approaches $+\infty$. ■

2.2 Stationary Set

In this subsection, we explicitly compute the stationary set defined page 34 for the potential n^2 constructed. To do so we compute the set M of trajectories issued from 0 which go back to 0.

Lemma II.8. *Let n^2 be the potential defined in (II.12) satisfying the structural hypothesis (II.14). In this case the set of trajectories which go back to 0 defined at the level of (II.10) is*

$$M = \left\{ (T_R, p, \eta), \text{ s.t. } \eta = -p = (r, \theta_1, \dots, \theta_{d-1}), \ r = \sqrt{2n^2(0)}, \ |\theta_1| \leq \theta_0 \right\},$$

where T_R is a positive constant characterized by $X(T_R, 0, (\sqrt{2n^2(0)}, 0, \dots, 0)) = 0$.

Proof of Lemma II.8. We consider $x(t)$ a Hamiltonian trajectory issued from 0 ($x(0) = 0$) on the zero-energy level ($x'(0) = p = (r, \theta_1, \dots, \theta_{d-1})$ in hyperspherical coordinates).

If $2\theta_0 \leq |\theta_1|$, the trajectory is a straight line which never enters B_p . It never goes back to 0.

We have to understand what happens when the trajectory reaches B_p . There are two cases. If it reaches $\{(r, \theta_1, \dots, \theta_d), \ R - 1 \leq r \leq R + 1, \ |\theta_1| \leq \theta_0\}$, since there is no contribution of the force in the orthoradial direction, it has to go back to 0. On the contrary, if it reaches $\{(r, \theta_1, \dots, \theta_d), \ R - 1 \leq r \leq R + 1, \ \theta_0 < |\theta_1| \leq 2\theta_0\}$ the action of the orthoradial force prevents the trajectory to go back to 0.

• **The trajectory reaches $\{(r, \theta_1, \dots, \theta_d), \ R - 1 \leq r \leq R + 1, \ |\theta_1| \leq \theta_0\}$** • We consider the trajectory $y(t)$ defined in hyperspherical coordinates:

$$y(t) = (r(t), \theta_1, \dots, \theta_{d-1}),$$

with $r(t)$ solution of the following ordinary equation $r'' = -\mu f'(r)$ with initial datas:

$$r(0) = 0, \quad r'(0) = \sqrt{2n^2(0)}.$$

Then y is solution of the Hamilton equation (II.9) and satisfies $y(0) = x(0)$, $x'(0) = y'(0)$. Thus $x = y$, the trajectory is radial. The radial trajectory $t \mapsto r(t)$ reaches the radial bump $\{R - 1 \leq r \leq R + 1\}$ at time t_e :

$$t_e = \inf \{t > 0, x(t) \in B_p\},$$

According to Corollary II.7, the trajectory has to go out of the radial bump $\{R - 1 \leq r \leq R + 1\}$. Either it leaves the bump at $r = R - 1$ or at $r = R + 1$. The case $r = R + 1$ is forbidden. Indeed by continuity, there would exist a time t_c such that $r(t_c) = R$ and so $x(t_c) \in B_\emptyset$. This is absurd. There exists a time $t_s > t_e$ such that the trajectory leaves the bump B_p ($r(t_s) = R - 1$).

By conservation of energy on the zero-energy level and due to the fact that the trajectory is radial we deduce that the trajectory has to go back to 0 with a speed η satisfying $\eta = -p$. We denote by $T_R > 0$ the time when the trajectory goes back to 0. It only depends on the characteristics of the potential, not on $\theta_1, \dots, \theta_{d-1}$.

• **The trajectory reaches** $\{(r, \theta_1, \dots, \theta_d), \quad R - 1 \leq r \leq R + 1, \quad \theta_0 < |\theta_1| \leq 2\theta_0\}$ • We first assume that $d = 2$. Using some symmetries of the system we can generalise the result to $d \geq 3$.

◦ In dimension $d = 2$ Hamilton equations (II.9) can be written in hyperspherical coordinates:

$$\begin{cases} r'' - r(\theta_1')^2 = -\mu f'(r)g(\theta_1), \\ 2r'\theta_1' + r\theta_1'' = -\mu \frac{f(r)}{r} g'(\theta_1). \end{cases}$$

Examining the second equation: $(r^2\theta_1')' = 2rr'\theta_1' + r^2\theta_1'' = -\mu f(r)g(\theta_1)$, we get when $r \neq 0$,

$$\theta_1'(t) = -\frac{\mu}{r^2(t)} \int_{t_e}^t f(r(s))g'(\theta_1(s))ds, \quad (\text{II.21})$$

where t_e characterizes the time when the trajectory enters B_p (defined in the preceding case).

By assumption, $\theta(t_e) > \theta_0$, and since the Hamiltonian trajectories are continuous, there exists $\varepsilon > 0$ such that for any time $t \in [t_e, t_e + \varepsilon]$ we have $\theta(t) > \theta_0$.

We recall that the function f is non-negative and that for all $\theta \geq \theta_0$ we have $g'(\theta) \geq 0$ by construction. In particular it implies that for all time $t \geq t_e$, $\theta(t) > \theta_0$. If not, it means that there exists t_c such that $\theta(t_c) = \theta_0$ and for all time $t \in [t_e, t_c]$, $\theta(t) > \theta_0$. Thus there exists $t_d \in]0, t_c[$ such that $\theta_1'(t_d) < 0$, i.e.:

$$\int_{t_e}^{t_c} f(r(s))g'(\theta_1(s))ds \geq 0.$$

On $[t_e, t_c]$, the function $t \mapsto f(r(t))g'(\theta_1(t))$ is non-positive. That is absurd. Thus for all $t \in [t_e, +\infty[$, $\theta(t) > \theta_0$. One consequence is that on $[t_e, +\infty[$ the application $t \mapsto \theta(t)$ is increasing (see (II.21)). Furthermore for all $t \in [t_e, +\infty[$ we have $\theta(t) \geq \theta(t_e) > \theta_0$.

To prove that the trajectory never goes back to 0 we proceed by contradiction. We assume that there exists $t_0 > 0$ such that $x(t_0) = 0$. We denote by t_s the last time when the trajectory is in B_p (we recall from Corollary II.7 that it exists):

$$t_s := \sup\{t > t_e, \quad x(t) \in B_p\}.$$

For all time $t_r \in [t_e, t_s]$, we have $r(t_r) \neq 0$. If not, since the trajectory is a straight line outside B_p , it means that for all time $t \geq t_r$ either $x(t) \notin B_p$ or for all time $t \leq t_r$ $x(t) \notin B_p$ which is absurd by construction of t_s .

For the same reason (straight line trajectory) the trajectory goes back to 0 in t_0 if and only if $\theta'(t_s) = 0$:

$$\theta'_1(t_s) = -\frac{\mu}{r^2(t_s)} \int_{t_e}^{t_s} f(r(s))g'(\theta_1(s))ds = 0.$$

Since for all time $t \in [t_e, t_s]$, $\theta(t) > \theta_0$, we deduce that $t \mapsto f(r(t))g'(\theta_1(t))$ has a constant sign on $[t_e, t_s]$. It comes that for all time $t \in [t_e, t_s]$ $f(r(s)) = 0$ or $g'(\theta(s)) = 0$. In particular the time $t_e + \varepsilon$ satisfies

$$f(r(t_e + \varepsilon))g'(\theta(t_e + \varepsilon)) = 0.$$

That is absurd. Finally $\theta'(t_s) \neq 0$ and the trajectory never goes back to 0.

◦ When the dimension d is larger than 3 we use the invariance of n^2 under the action of $\mathbb{O}_{d,1}(\mathbb{R})$. Let $p = (r, \theta_1, \dots, \theta_{d-1})$. There exists $A \in \mathbb{O}_{d,1}(\mathbb{R})$ such that $Ap = (\sqrt{2n^2(0)}, \theta_1, 0, \dots, 0)$.

We denote by $(r(t), \theta_1(t))$ the solution (in hyperspherical coordinates) of the Hamilton equation (II.9) with initial data $(\sqrt{2n^2(0)}, \theta_1)$ in dimension 2.

We set $y(t) = A^{-1}(r(t), \theta_1(t), 0, \dots, 0)$. Then y is solution of Hamilton equation (II.9) with initial data $y(0) = 0$, $y'(0) = p$. By uniqueness, $x = y$. We just prove that for all time $t > 0$, $r(t) \neq 0$. The trajectory can not go back to 0. ■

2.3 Transversality condition: H_2

In this subsection, we assume that the condition (II.14) is satisfied. We prove that the transversality condition H_2 (defined page 34) is satisfied for the potential n^2 constructed by (II.12). To that end, we explicitly compute the tangent plane $T_m M_X$ (defined in (II.17)) and the kernel $\text{Ker } D^2\psi|_m$ where $\text{Ker } D^2\psi|_m$ is defined page 50. In particular, it requires to compute the B and D components of the linear flow.

We start by computing them for a particular point $m_0 \in \overset{\circ}{M}_X$:

$$m_0 := (0, 0, p_0, -p_0, 0, p_0, T_R), \quad \text{where } p_0 := (\sqrt{2n^2(0)}, 0, \dots, 0).$$

We use particular symmetries of the potential to extend property H_2 to any $m \in \overset{\circ}{M}_X$.

2.3.1 Computation for m_0

The computation of $T_{m_0}M_X$ is rather easy:

Lemma II.9. *The tangent plane at M_X in m_0 is:*

$$T_{m_0}M_X = \{(X, Y, \Xi, H, Q, P, T), \text{ s.t. } X = Y = Q = T = 0, \Xi = P = -H, P.p_0 = 0\}.$$

Proof of Lemma II.9. We recall that M_X is defined thanks to the submanifold M (II.17). We have already computed M at the level of Lemma II.8. A mere computation leads to the result. \blacksquare

To determine $\text{Ker } D^2\psi|_{m_0}$ the first step is to compute the matrices B and D components of the linearized flow defined at the level of (II.25).

Lemma II.10. *Let n^2 be the potential constructed according to (II.12). We have:*

$$D(T_R, 0, p_0) := \frac{\partial \Xi}{\partial \xi}(T_R, 0, p_0) = -I_d, \quad B(T_R, 0, p_0) := \frac{\partial X}{\partial \xi}(T_R, 0, p_0) = \begin{pmatrix} b_{11} & 0 \\ 0 & O_{d-1} \end{pmatrix}, \quad (\text{II.22})$$

where I_d is the identity matrix, $b_{11} \in \mathbb{R}$ and O_{d-1} is a square matrix of dimension $d-1$ equal to 0.

Proof of Lemma II.10. We consider $x_0(t, 0, p) = (x_0^1(t, 0, p), \dots, x_0^d(t, 0, p))$ the solution of (II.9) with initial data $x_0(0, 0, p) = 0$ and $x'_0(0, 0, p) = p$.

We recall that the potential n^2 is invariant under the action of $\mathbb{O}_{d,1}(\mathbb{R}^d)$. Thus we first compute the component of D and B invariant under $\mathbb{O}_{d,1}(\mathbb{R}^d)$ that is to say their first column. Then we consider the orthogonal of e_1 . Those components are more difficult to compute. We use the symmetries again and a perturbation argument to determine the last columns of D and B .

• **Computation of $\frac{\partial \Xi}{\partial \xi_1}(T_R, 0, p_0)$ and $\frac{\partial X}{\partial \xi_1}(T_R, 0, p_0)$** • We start with $\frac{\partial \Xi_j}{\partial \xi_1}(T_R, 0, p_0)$ for $j \geq 2$.

$$\frac{\partial \Xi_j}{\partial \xi_1}(T_R, 0, p_0) = \lim_{\varepsilon \rightarrow 0} \frac{\Xi_j(T_R, 0, (\sqrt{2n^2(0)} + \varepsilon, 0, \dots, 0)) - \Xi_j(T_R, 0, (\sqrt{2n^2(0)}, 0, \dots, 0))}{\varepsilon}.$$

Nevertheless since the trajectory is radial we deduce that

$$\Xi_j(T_R, 0, (\sqrt{2n^2(0)} + \varepsilon, 0, \dots, 0)) = \Xi_j(T_R, 0, (\sqrt{2n^2(0)}, 0, \dots, 0)) = 0, \quad \forall j \geq 2.$$

Finally, $\frac{\partial \Xi_j}{\partial \xi_1}(T_R, 0, p_0) = 0, \forall j \geq 2$. A similar argument provides $\frac{\partial X_j}{\partial \xi_1}(T_R, 0, p_0) = 0, \forall j \geq 2$. There remains to determine the first coefficient of D : $\frac{\partial \Xi_1}{\partial \xi_1}(T_R, 0, p_0)$. Since the trajectory is radial and by conservation of the energy we have for ε small enough:

$$\Xi_1(T_R, 0, \sqrt{2n^2(0)} + \varepsilon) = -(\sqrt{2n^2(0)} + \varepsilon) \quad \text{and} \quad \Xi_1(T_R, 0, \sqrt{2n^2(0)}) = -\sqrt{2n^2(0)}.$$

Thus,

$$d_{11} := \lim_{\varepsilon \rightarrow 0^+} \frac{\Xi(T_R, 0, \sqrt{2n^2(0)} + \varepsilon) - \Xi(T_R, 0, \sqrt{2n^2(0)})}{\varepsilon} = -1.$$

• **Computation of $\frac{\partial \Xi(T_R, 0, p_0)}{\partial \xi_j}$ and $\frac{\partial X(T_R, 0, p_0)}{\partial \xi_j}$ ($j \geq 2$)** • Considering the symmetries, we assume $j = 2$. The other components are determined with exactly the same argument. We perturb the initial speed of ε along the direction e_2 (see Figure II.6).

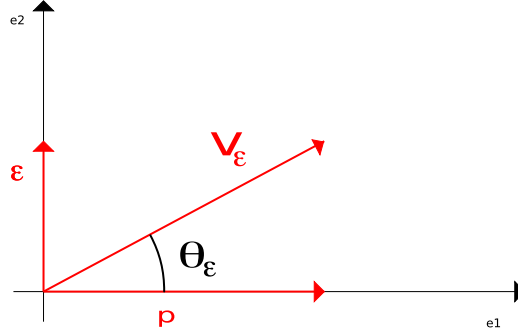


Figure II.6: Perturbation of the initial speed

Let X_ε be the solution of the perturbed problem:

$$\begin{cases} X''_\varepsilon = \nabla n^2(X_\varepsilon), & X_\varepsilon(0) = 0, & X'_\varepsilon(0) = p_0 + \varepsilon e_2. \end{cases}$$

We expand X_ε with respect to ε and obtain $X_\varepsilon(t) = X_0(t) + \varepsilon X_1(t) + \dots$. With these notations we have $X_1(t) = \frac{DX}{D\xi_2}(t)$ and $X'_1(t) = \frac{D\Xi}{D\xi_2}(t)$. To obtain the expansion in ε , we try to go back to the previous case ($j = 1$) by making some change of variables. For ε small enough, the trajectory is radial along the direction $X'_\varepsilon(0)$. Let $(\tilde{e}_1, \dots, \tilde{e}_d)$ be a new basis defined by $\tilde{e}_j := O_\varepsilon e_j$, with

$$O_\varepsilon := \begin{pmatrix} \cos(\theta_\varepsilon) & -\sin(\theta_\varepsilon) & 0 & \dots & 0 \\ \sin(\theta_\varepsilon) & \cos(\theta_\varepsilon) & 0 & \dots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & I_{d-2} & \\ 0 & 0 & & & \end{pmatrix}, \quad \cos(\theta_\varepsilon) = \frac{p_0}{p_0^2 + \varepsilon^2}, \quad \sin(\theta_\varepsilon) = \frac{\varepsilon}{p_0^2 + \varepsilon^2}.$$

Let $\widetilde{X}_\varepsilon$ be the coordinates of X_ε in $(\tilde{e}_1, \dots, \tilde{e}_d)$. Since $O_\varepsilon^{-1} \nabla n^2(X_\varepsilon) = \nabla n^2(\widetilde{X}_\varepsilon)$, we clearly have

$$\begin{cases} \widetilde{X}_\varepsilon''(t) = \nabla n^2(\widetilde{X}_\varepsilon(t)), & \widetilde{X}_\varepsilon(0) = 0, & \widetilde{X}_\varepsilon'(0) = (\sqrt{\varepsilon^2 + p_0^2}, 0, \dots, 0) = p_0 + O(\varepsilon^2). \end{cases}$$

Then, it is clear that $\widetilde{X}_\varepsilon(t) = \widetilde{X}_0(t) + O(\varepsilon^2)$. We determine X_0 and X_1 thanks to \widetilde{X}_0 and \widetilde{X}_1 :

$$X_0(t) + \varepsilon X_1(t) = O_\varepsilon \left(\widetilde{X}_0(t) + O(\varepsilon^2) \right) = (I_d + \varepsilon E + O(\varepsilon^2))(\widetilde{X}_0(t) + O(\varepsilon^2)),$$

with

$$E := \begin{pmatrix} 0 & -\frac{1}{p_0} & 0 & \dots & 0 \\ \frac{1}{p_0} & 0 & 0 & \dots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & I_{d-2} & \\ 0 & 0 & & & \end{pmatrix}.$$

In other words:

$$\forall t \in \mathbb{R}, \quad X_0(t) = \widetilde{X}_0(t) \quad \text{and} \quad X_1(t) = E\widetilde{X}_0(t).$$

Since the Hamiltonian trajectory goes back to 0 at time T_R , we deduce

$$\frac{\partial X}{\partial \xi_2}(T_R, 0, p_0) = X_1(T_R) = E\widetilde{X}_0(T_R, 0, p_0) = E \times 0 = 0.$$

In the same way,

$$\begin{aligned} \frac{\partial \Xi}{\partial \xi_2}(T_R, 0, p_0) &= X'_1(T_R) = E\widetilde{X}'_0(T_R) = {}^t \left(-\frac{x'^2_{0,R}(T_R)}{p_0}, \frac{x'^1_{0,R}(T_R)}{p_0}, x'^3_{0,R}(T_R), \dots, x'^d_{0,R}(T_R) \right), \\ &= {}^t(0, 1, 0, \dots, 0). \end{aligned}$$

Now the columns of B and D (for $j \geq 3$) are determined by defining a smooth transformation again. It finally leads to (II.22). \blacksquare

At this stage, we can identify the kernel of $D^2\psi$, computed in (II.31), with the tangent plane.

Corollary II.11. $\text{Ker } D^2\psi|_{m_0} = T_{m_0}M_X$.

Proof of Corollary II.11. According to (II.31) below, we have:

$$\begin{aligned} \text{Ker}(D^2\psi|_m) &= \{(X, Y, \Xi, H, Q, P, T), X = Y = Q = 0, \\ &\quad \Xi = P, \eta^T H = 0, B_{T_R}P + T\eta = 0, -H + D_{T_R}P + T\nabla n^2(0) = 0\}. \end{aligned}$$

Since $\eta = -p_0$, H has to satisfy $H = (0, H_2, \dots, H_d)$ (in Cartesian coordinates). Since $\nabla n^2(0) = 0$, we deduce that $D_{T_R}P = H$. According to Lemma II.10, we deduce that $H = -P$. Finally, $B_{T_R}P = 0$ so $T = 0$. Thus,

$$\text{Ker } D^2\psi|_{m_0} = \{(X, Y, \Xi, H, Q, P, T), X = Y = Q = T = 0, P = \Xi = -H, P.p_0 = 0\}.$$

From Lemma II.9, we get $\text{Ker } D^2\psi|_{m_0} = T_{m_0}M_X$. This completes the proof. \blacksquare

2.3.2 Generalisation

In this subsection, we generalize Corollary II.11 to any point $m \in \overset{\circ}{M}_X$.

Lemma II.12. $\forall m \in \overset{\circ}{M}_X, T_m M_X = \text{Ker } D^2\psi|_m$.

Proof of Lemma II.12. The idea is to use a family of transformations which let $\overset{\circ}{M}_X$ and n^2 invariant (in a sense we define later). Then we transport (thanks to those transformations) the equality $\text{Ker } D^2\psi|_{m_0} = T_{m_0}M_X$ to any $m \in \overset{\circ}{M}_X$.

Family of transformations. Let $m \in \overset{\circ}{M}_X$. We write $m = (0, 0, p, -p, 0, p, T_R)$ for some $p \in \sqrt{2n^2(0)}\mathbb{S}^{d-1}$. Thus, there exists $R_p \in \mathcal{O}(\mathbb{R}^d)$ such that $R_p(p) = p_0$. We define the map $\tilde{R}_m : \mathbb{R}^{6d+1} \longrightarrow \mathbb{R}^{6d+1}$ such that

$$\tilde{R}_m(x, y, \xi, \eta, q, p, t) = (R_p(x), R_p(y), R_p(\xi), R_p(\eta), R_p(q), R_p(p), t).$$

By construction $\tilde{R}_m(m) = m_0$.

Action on the tangent place. The set

$$M_X^r := \{(x, y, \xi, \eta, q, p, t) \in \mathbb{R}^{6d+1} \text{ s.t. } x = y = q = 0, \xi = p = -\eta, t = T_R, |p| = \sqrt{2n^2(0)}\}$$

is invariant under the action of \tilde{R}_m . In other words it satisfies: $\forall n \in M_X^r, \tilde{R}_m n \in M_X^r$. Thus, as \tilde{R}_m is continuous, there exists U a neighborhood of m in $\overset{\circ}{M}_X$ (perhaps very small) such that $U_0 := \tilde{R}_m U \subset \overset{\circ}{M}_X$. The application \tilde{R}_m defines a linear map from U to U_0 such that $\tilde{R}_m(m) = m_0$. Thus, $\tilde{R}_m(T_m M_X) = T_{m_0} M_X$.

Action on the kernel. We hope the action of \tilde{R}_m on $\text{Ker } D^2\psi|_m$ to be the same. Thus we compute the set $\tilde{R}_m(\text{Ker}(D^2\psi|_m))$:

$$\begin{aligned} \tilde{R}_m(\text{Ker}(D^2\psi|_m)) &= \{(R_p X, R_p Y, R_p \Xi, R_p H, R_p Q, R_p P, T), \text{ s.t. } X = Y = Q = 0, \\ &\quad p.H = 0, B_{T_R}(p)P + Tp = 0, D_{T_R}(m)P = H\}, \\ &= \{(X, Y, \Xi, H, Q, P, T), \text{ s.t. } X = Y = Q = 0, \\ &\quad p.R_p^{-1}H = 0, B_{T_R}(p)R_p^{-1}P + Tp = 0, D_{T_R}(p)R_p^{-1}P = R_p^{-1}H\}. \\ &= \{(X, Y, \Xi, H, Q, P, T), \text{ s.t. } X = Y = Q = 0, \\ &\quad p_0.H = 0, R_p B_{T_R}(p)R_p^{-1}P + Tp_0 = 0, R_p D_{T_R}(p)R_p^{-1}P = H\}. \end{aligned}$$

Furthermore using the symmetries we have:

$$R_p B_{T_R}(p)R_p^{-1} = B_{T_R}(p_0), \quad R_p D_{T_R}(p)R_p^{-1} = D_{T_R}(p_0). \quad (\text{II.23})$$

Indeed since the potential is radial on $\Pi_x U$ (the projection on the space variable x of U), we have $n^2(R_p x) = n^2(x)$. We deduce,

$$R_p \frac{D^2 n^2}{Dx^2}(x) R_p^{-1} = \frac{D^2 n^2}{Dx^2}(R_p x).$$

Applying this at $x = X(t, 0, p)$ leads to

$$R_p \frac{D^2 n^2}{Dx^2}(X(t, 0, p)) R_p^{-1} = \frac{D^2 n^2}{Dx^2}(R_p X(t, 0, p)) = \frac{D^2 n^2}{Dx^2}(X(t, 0, p_0)).$$

Thus the variables $R_p B_t(p)R_p^{-1}$ and $R_p D_{T_R}(p)R_p^{-1}$ satisfy:

$$\partial_t R_p B_t(p)R_p^{-1} = R_p D_t(p)R_p^{-1},$$

and

$$\partial_t R_p D_t(p) R_p^{-1} = R_p D^2 n^2(X(t, 0, p)) R_p^{-1} R_p B_t(p) R_p^{-1} = D^2 n^2(X(t, 0, p_0)) R_p B_t(p) R_p^{-1},$$

together with the initial data: $R_p B_0(p) R_p^{-1} = 0$ and $R_p D_t(p) R_p^{-1} = I_d$. By uniqueness of the linearized flow we obtain that $R_p B_t(p) R_p^{-1} = B_t(p_0)$ and $R_p D_t(p) R_p^{-1} = B_t(p_0)$. Particularizing $t = T_R$, it proves (II.23).

Finally, $\tilde{R}_m(\text{Ker}(D^2 \psi|_{m_0})) = \text{Ker} D^2 \psi|_{m_0}$. ■

3 Proof of the convergence

This section is dedicated to the computation of the contribution of

$$J_\varepsilon := \frac{1}{\varepsilon} \int_\theta^{T_1} \left(1 - \chi\left(\frac{t}{\theta}\right) \right) e^{-\alpha_\varepsilon t} \langle U_\varepsilon(t) \chi_\delta(H_\varepsilon) S_\varepsilon, \phi_\varepsilon \rangle dt.$$

Here we consider n^2 constructed in (II.12) such that (II.14) is satisfied.

The first step is to write J_ε as an oscillatory integral in order to apply a stationary phase argument. We briefly recall how to obtain it with the help of the wave packet theorem due to M. Combes and D. Robert (see [13]). Nevertheless since it is detailed in [13], we skip some estimates.

In a second part, we apply the stationary phase theorem to prove assertion *iii*) in the main Theorem II.1

3.1 Notations - Around the linearized flow

In this section we introduce the definition of the linearized flow. We also give some notations.

Let $\varphi(t, x, \xi) = (X(t, x, \xi), \Xi(t, x, \xi))$ denote the flow. The linearized flow, written F , is

$$F(t, x, \xi) = \frac{D\varphi(t, x, \xi)}{D(x, \xi)} := \begin{pmatrix} A(t, x, \xi) & B(t, x, \xi) \\ C(t, x, \xi) & D(t, x, \xi) \end{pmatrix},$$

where $A(t)$, $B(t)$, $C(t)$, $D(t)$ are by definition

$$\begin{aligned} A(t, x, \xi) &= \frac{DX(t, x, \xi)}{Dx}, & B(t, x, \xi) &= \frac{DX(t, x, \xi)}{D\xi}, \\ C(t, x, \xi) &= \frac{D\Xi(t, x, \xi)}{Dx}, & D(t, x, \xi) &= \frac{D\Xi(t, x, \xi)}{D\xi}. \end{aligned}$$

The linearisation of (II.9) leads to write:

$$\begin{cases} \frac{\partial}{\partial t} A(t, x, \xi) = C(t, x, \xi), & A(0, x, \xi) = Id, \\ \frac{\partial}{\partial t} C(t, x, \xi) = \frac{D^2 n^2}{Dx^2}(X(t, x, \xi)) A(t, x, \xi), & C(0, x, \xi) = 0, \end{cases} \quad (\text{II.24})$$

and

$$\begin{cases} \frac{\partial}{\partial t} B(t, x, \xi) = D(t, x, \xi), & B(0, x, \xi) = Id, \\ \frac{\partial}{\partial t} D(t, x, \xi) = \frac{D^2 n^2}{Dx^2}(X(t, x, \xi)) B(t, x, \xi), & D(0, x, \xi) = 0. \end{cases} \quad (\text{II.25})$$

Finally, we introduce the matrix $\Gamma(t, q, p)$:

$$\Gamma(t, q, p) = (C(t, q, p) + iD(t, q, p))(A(t, q, p) + iB(t, q, p))^{-1}.$$

3.2 A wave packet approach

In this subsection, we recall how to obtain an oscillatory integral thanks to a wave packet approach. In the sequel we use the shorthand notation (q_t, p_t) as the Hamiltonian trajectory (II.9) with the initial data (q, p) :

$$(q_t, p_t) := (X(t, q, p), \Xi(t, q, p)).$$

Lemma II.13. *Select an integer $N \in \mathbb{N}$ and a time $T_1 \in \mathbb{R}_+^*$. J_ε can be written as:*

$$J_\varepsilon = \frac{1}{\varepsilon^{(5d+2)/2}} \int_\theta^{T_1} \int_{\mathbb{R}^{6d}} e^{i\psi(t, X)} a_N(t, X) dt dX + O_{T_1, \delta}(\varepsilon^N),$$

with $X = (q, p, x, y, \xi, \eta)$. The amplitude a_N is defined in (II.28) and the complex phase ψ is specified in (II.29). The amplitude a_N depends on two truncations χ_0 and χ_1 such that:

- $\chi_0 \in C_0^\infty(\mathbb{R}^{2d})$ fulfils

$$\begin{aligned} \text{supp } \chi_0(q, p) &\subset \{|q| \leq 2\delta\} \cup \{|p^2/2 - n^2(q)| \leq 2\delta\}, \\ \chi_0 &\equiv 1 \text{ on } \{|q| \leq 3/2\delta\} \cup \{|p^2/2 - n^2(q)| \leq 3/2\delta\}, \end{aligned}$$

- $\chi_1 \in C_0^\infty(\mathbb{R}^{2d})$, $\chi_1 \equiv 1$ close to $(0, 0)$.

Steps of the proof of Lemma II.13. Writing $\tilde{\chi}_\delta(t) := e^{-\alpha_\varepsilon t}(1 - \chi)(t/\theta)$, the integral J_ε is reduced to

$$J_\varepsilon = i/\varepsilon \int_\theta^{T_1} \tilde{\chi}_\delta(t) \langle \chi_\delta(H_\varepsilon) S_\varepsilon, U_\varepsilon(-t) \phi_\varepsilon \rangle dt.$$

To obtain estimations, we need to understand the way $U_\varepsilon(-t)$ acts on ϕ_ε . To this end, we define the Gaussian wave packet:

$$\varphi_{q,p}^\varepsilon(x, \xi) := \frac{1}{(\pi\varepsilon)^{d/4}} \exp\left(\frac{i}{\varepsilon} p \cdot \left(x - \frac{q}{2}\right)\right) \exp\left(-\frac{(x - q)^2}{2\varepsilon}\right).$$

The computation of the action on the wave packet is easier than on an unspecified function. Furthermore, the family $\{\varphi_{q,p}^\varepsilon\}_{q,p \in \mathbb{R}}$ forms an overcomplete basis of $L^2(\mathbb{R}^{2d})$. For that reason, we write

$$\begin{aligned} \langle \chi_\delta(H_\varepsilon) S_\varepsilon, U_\varepsilon(-t) \phi_\varepsilon \rangle &= \frac{1}{(2\pi\varepsilon)^d} \int_{\mathbb{R}^{2d}} \langle \chi_\delta(H_\varepsilon) S_\varepsilon, \varphi_{q,p}^\varepsilon \rangle \langle \varphi_{q,p}^\varepsilon, U_\varepsilon(-t) \phi_\varepsilon \rangle dq dp, \\ &= \frac{1}{(2\pi\varepsilon)^d} \int_{\mathbb{R}^{2d}} \langle \chi_\delta(H_\varepsilon) S_\varepsilon, \varphi_{q,p}^\varepsilon \rangle \langle U_\varepsilon(t) \varphi_{q,p}^\varepsilon, \phi_\varepsilon \rangle dq dp. \end{aligned}$$

Using some functional calculus of Helffer and Robert for pseudo-differential operators [25] together with some Wigner transforms, we can replace the term $\chi_\delta(H_\varepsilon)$ by some truncation function χ_0 (satisfying the assumptions of the Lemma) up to a term of order $O_{\delta, T_1}(\varepsilon^\infty)$:

$$J_\varepsilon = \frac{i}{\varepsilon(2\pi\varepsilon)^d} \int_\theta^{T_1} \int_{\mathbb{R}^{2d}} \tilde{\chi}_\delta(t) \chi_0(q, p) \langle S_\varepsilon, \varphi_{q,p}^\varepsilon \rangle \langle U_\varepsilon(t) \varphi_{q,p}^\varepsilon, \phi_\varepsilon \rangle dq dp dt + O_{\delta, T_1}(\varepsilon^\infty).$$

Then, the action of U_ε on $\varphi_{q,p}^\varepsilon$ is computed according to the following theorem (see [13, 34, 35]):

Theorem II.14. *Let $n^2(x)$ a long-range potential. There exists a family of functions $\{p_{k,j}(t, q, p, x)\}_{(k,j) \in \mathbb{N}^2}$ that are polynomials of degree at most k in the variable $x \in \mathbb{R}^d$, with coefficients depending on t, q, p , such that for any $\varepsilon \leq 1$, the following estimate holds: for any T_1 , for any integer N , there exists C_{N,T_1} such that for any $t \in [0, T_1]$:*

$$\left\| U_\varepsilon(t) \varphi_{q,p}^\varepsilon - \exp\left(\frac{i}{\varepsilon} \delta(t, q, p)\right) T_\varepsilon \Delta_\varepsilon \mathcal{Q}_N(t, q, p, x) M(F(t, q, p)) (\pi^{-d/4} \exp(-x^2/2)) \right\|_{L^2(\mathbb{R}^d)} \leq C_{N,T_1} \varepsilon^N, \quad (\text{II.26})$$

where

$$\begin{cases} \mathcal{Q}_N(t, q, p, x) = 1 + \sum_{(k,j) \in I_N} \varepsilon^{\frac{k}{2}-j} p_{k,j}(t, q, p, x), \\ I_N = \{1 \leq j \leq 2N-1, 1 \leq k-2j \leq 2N-1, K \geq 3j\}, \end{cases} \quad (\text{II.27})$$

and

- Δ_ε is a dilatation operator:

$$(\Delta_\varepsilon u)(x) = \varepsilon^{-d/4} u\left(\frac{x}{\sqrt{\varepsilon}}\right),$$

- $T_\varepsilon(q_t, p_t)$ is the translation in phase-space operator (or Weyl-Heisenberg operator) :

$$(T_\varepsilon(q_t, p_t)u)(x) = \exp\left(\frac{i}{\varepsilon} p_t \cdot \left(x - \frac{q_t}{2}\right)\right) u(x - q_t),$$

- $\delta(t, q, p)$ is defined as:

$$\delta(t, q, p) = \int_0^t \left(\frac{p_s^2}{2} + n^2(q_s) \right) ds - \frac{q_t \cdot p_t - p \cdot q}{2},$$

- $M(F(t, q, p))$ is the metaplectic operator associated with the symplectic matrix $F(t, q, p)$. It acts on the Gaussian as:

$$M(F(t, q, p)) \left(\exp\left(-\frac{x^2}{2}\right) \right) = \det(A(t, q, p) + iB(t, q, p))_c^{-1/2} \exp\left(i \frac{\Gamma(t, q, p) x \cdot x}{2}\right),$$

where the square root $\det(A(t, q, p) + iB(t, q, p))_c^{-1/2}$ is defined by continuity following the argument of the complex number $\det(A(t, q, p) + iB(t, q, p))$ starting from its value 1 at time $t=0$. Finally $\Gamma(t, q, p)$ is defined as:

$$\Gamma(t, q, p) = (C(t, q, p) + iD(t, q, p))(A(t, q, p) + iB(t, q, p))^{-1}.$$

■

At the end of the proof, we apply the above theorem and we transform J_ε accordingly.

At first, in order to prepare a stationary phase argument, we need some compactness on the variables of integration. To do so, we use the Parseval formula:

$$\langle S_\varepsilon, \varphi_{q,p} \rangle = \frac{1}{\varepsilon^{d/2}} \int_{\mathbb{R}_{x,\xi}^{2d}} e^{\frac{i}{\varepsilon} x \cdot \xi} \widehat{S}(\xi) \varphi_{q,p}(x) dx d\xi = \frac{1}{\varepsilon^{d/2}} \int_{\mathbb{R}_{x,\xi}^{2d}} \chi(x) e^{\frac{i}{\varepsilon} x \cdot \xi} \widehat{S}(\xi) \varphi_{q,p}(x) dx d\xi + o(\varepsilon^\infty),$$

for some truncation function χ equal to 1 close to the origin. The use of the Fourier transform seems technical. However to understand how the source radiates, we have to make the Fourier transform of S appear.

Secondly, we replace $U_\varepsilon \varphi_{q,p}^\varepsilon$ according to (II.26). We use the short-hand notation

$$P_N(t, q, p, x) := \frac{1}{\pi^{d/4}} \det(A(t, q, p) + iB(t, q, p))_c^{-1/2} \mathcal{Q}_N(t, q, p, x).$$

We use again a Parseval formula in the variable y (and still introduce some truncation function $\chi \equiv 1$ near the origin) to finally obtain:

$$J_\varepsilon = \frac{1}{\varepsilon^{(5d+2)/2}} \int_\theta^{T_1} \int_{\mathbb{R}^{6d}} e^{\frac{i}{\varepsilon} \psi(t, X)} a_N(t, X) dt dX + O_{T_1, \delta}(\varepsilon^N),$$

with the amplitude a_N defined as

$$a_N(t, X) := \widehat{S}(\xi) \widehat{\phi}^*(\eta) \chi_0(q, p) \chi_1(x, y) P_N\left(t, q, p, \frac{y - q_t}{\sqrt{\varepsilon}}\right), \quad (\text{II.28})$$

and

$$\begin{aligned} \psi(t, X) := \int_0^t \left(\frac{p_s^2}{2} + n^2(q_s) \right) ds - p \cdot (x - q) + p_t \cdot (y - q_t) \\ + x \cdot \xi - y \cdot \eta + i \frac{(x - q)^2}{2} + \frac{1}{2} \Gamma_t(x - q_t) \cdot (x - q_t). \end{aligned} \quad (\text{II.29})$$

For more information, the user can read [7]. This ends the proof of Lemma II.13. \blacksquare

3.3 A stationary phase argument - Proof of iii)

In this subsection, we perform a stationary phase method on the oscillating integral obtained in Subsection II.13. We recall that the stationary set M_X associated with ψ is the set of points

$$M_X := \left\{ (t, X) \in]0, +\infty[\times \mathbb{R}^{6d}, \text{ s.t. } \text{Im } \psi = 0 \text{ and } \nabla \psi = 0 \right\}.$$

We can explicitly compute it as well as $\text{Ker } D^2 \psi|_m$ for any $m \in M_X$.

Lemma II.15. *Consider n^2 a non-trapping potential:*

$$\bullet M_X := \left\{ (t, X) \in]0, +\infty[\times \mathbb{R}^{6d} \text{ s.t. } x = y = q = 0, \xi = p, (p, \eta, t) \in M \right\} \quad (\text{II.30})$$

$$\begin{aligned} \bullet \text{Ker}(D^2 \psi|_m) = \left\{ (T, X, Y, \Xi, H, Q, P) \in]0, +\infty[\times \mathbb{R}^{6d}, X = Y = Q = 0, \right. \\ \left. \Xi = P, \eta^T H = 0, B_t P + T \eta = 0, -H + D_t P + T \nabla n^2(0) = 0 \right\}, \end{aligned} \quad (\text{II.31})$$

where A_t B_t , etc. is a short-hand notation to mean $A(t)$, $B(t)$, etc. the linearized flow.

Proof of Lemma II.15. A mere computation of $\text{Im } \psi$ and $\nabla \psi$ allows to write (II.30). Then differentiating $\nabla \psi$ once allows to write (II.31). For more details, the reader can check [7]. \blacksquare

In the case of the potential n^2 constructed (according to (II.12)), we have already specified those two sets according to Lemma II.8 for the computation of M_X and Lemma II.12 for the computation of the kernel. We can now perform a stationary phase method.

Lemma II.16. *Let n^2 be the potential constructed according to II.12. Select a source $S \in \mathcal{S}(\mathbb{R}^d)$ such that $\text{supp}(\widehat{S}) \subset \{t.\eta \text{ s.t. } \eta \in I_{\theta_0}, t > 0\}$. Then,*

$$\forall \phi \in \mathcal{S}(\mathbb{R}^d), \quad |J_\varepsilon - L_\varepsilon(\phi)| = O(\sqrt{\varepsilon}).$$

L_ε can be written as:

$$L_\varepsilon(\phi) = C_{T_R, d} e \left(\frac{i}{\varepsilon} \int_0^{T_R} \left(\frac{\Xi(s, 0, p_0)^2}{2} + n^2(X(s, 0, p_0)) \right) ds \right) \int_{I_{\theta_0}} \widehat{S}(p) \widehat{\phi}^*(-p) d\lambda(p).$$

with $C_{T_R, d}$ a constant different from 0 (defined page 54).

Proof of Lemma II.16. Out of the stationary set, the oscillatory integral is of order $O(\varepsilon^\infty)$ so $J_\varepsilon = O_{T, \delta_1}(\varepsilon^N)$ (for some integer N chosen later). On the stationary set and near the support of a_N (denoted $\text{supp } a_N$) the stationary set associated with ψ is a submanifold of codimension $k = 6d + 1 - (d - 1) = 5d + 2$. Indeed, thanks to the hypothesis on the support of S , we have $\text{supp } a_N \cap \partial M_X = \emptyset$.

Writing $p = (r, \theta_1, \dots, \theta_{d-1})$ in hyper spherical coordinates, we define the application:

$$\begin{aligned} \gamma : \mathbb{R}^{6d+1} \cap \text{supp } a &\longrightarrow \mathbb{R}^{6d+1} \\ (t, x, y, \xi, \eta, q, p) &\longmapsto \underbrace{(t - T_R, x, y, \xi - p, \eta + p, q, r - \sqrt{2n^2(0)})}_{=: \alpha}, \underbrace{\theta_1, \dots, \theta_{d-1}}_{=: \beta} \end{aligned}$$

The map γ is a C^∞ -diffeomorphism from $\text{supp } a_N$ into its range. Furthermore,

$$(t, X) \in M_X \cap \text{supp } a_N \iff \alpha = 0.$$

In some way those new coordinates are more suited for the stationary set M_X associated with ψ . Making the change of variables $(t, X) = \gamma^{-1}(\alpha, \beta)$ in the integral defining J_ε we have:

$$\begin{aligned} J_\varepsilon = \frac{1}{\varepsilon^{(5d+2)/2}} \int_{\gamma(\text{supp } a_N)} e^{\frac{i}{\varepsilon} \psi \circ \gamma^{-1}(\alpha, \beta)} \left(\widehat{S}(\cdot) \widehat{\phi}^*(\cdot) P_N \left(\cdot, \cdot, \cdot, \frac{\cdot}{\sqrt{\varepsilon}} \right) \right) \circ \gamma^{-1}(\alpha, \beta) \chi_3(\alpha, \beta) d\alpha d\beta \\ + O_{\delta, T_1}(\varepsilon^N), \end{aligned}$$

where χ_3 is some truncation function. Since for all point $m \in M_X \cap \text{supp } a_N$ we have $\text{Ker}(D^2\psi|_m) = T_m M_X$ (Lemma II.12), the function $D^2\psi$ is non-degenerate in the normal direction of M_X . It means that

$$\det \left(\frac{D^2\psi \circ \gamma^{-1}}{D\alpha^2}(0, \beta) \right) \neq 0.$$

Furthermore, the projection of $\gamma(\text{supp } a_N)$ onto the space variable β is the set $\Pi_\theta I_{\theta_0}$ defined as

$$\Pi_\theta I_{\theta_0} := \{(\theta_1, \dots, \theta_{d-1}), \quad \theta_1 \in]-\theta_0, \theta_0[, \quad \theta_i \in]-\pi, \pi[, \quad \forall i \geq 2\}.$$

We can apply the stationary phase theorem. Remember that the codimension of the stationary set M_X associated with ψ is $k = 5d + 2$. For any integer L there exists a sequence $(Q_{2l}(\partial))_{l \in \{0, \dots, L\}}$ of operators of order $2l$ such that:

$$\begin{aligned} J_\varepsilon &= \int_{\Pi_\theta I_{\theta_0}} \exp\left(\frac{i}{\varepsilon} \psi \circ \gamma^{-1}(0, \beta)\right) \left(\left(Q_0(\cdot) \widehat{S}(\cdot) \widehat{\phi}^*(\cdot) P_N \left(\cdot, \cdot, \cdot, \frac{\cdot}{\sqrt{\varepsilon}} \right) \right) \circ \gamma^{-1} \chi_3 \right) (0, \beta) d\beta \\ &+ \int_{\Pi_\theta I_{\theta_0}} \exp\left(\frac{i}{\varepsilon} \psi \circ \gamma^{-1}(0, \beta)\right) \sum_{l=1}^L \varepsilon^l Q_{2l}(\partial) \left(\left(\widehat{S}(\cdot) \widehat{\phi}^*(\cdot) P_N \left(\cdot, \cdot, \cdot, \frac{\cdot}{\sqrt{\varepsilon}} \right) \right) \circ \gamma^{-1} \chi_3 \right) (0, \beta) d\beta \\ &+ O\left(\varepsilon^{L+1} \sup_{k \leq 2L+d+3} \left\| \partial_{(\alpha, \beta)}^k \left(\widehat{S}(\cdot) \widehat{\phi}^*(\cdot) P_N \left(\cdot, \cdot, \cdot, \frac{\cdot}{\sqrt{\varepsilon}} \right) \circ \gamma^{-1} \chi_3 \right) \right\| \right) + O_{\delta, T_1}(\varepsilon^N). \quad (\text{II.32}) \end{aligned}$$

We compute the several contritubions.

• **Contribution of the last term** • We look at the last term in the variable (t, X) instead of the variable (α, β) . What can be dangerous in the last term is the presence of terms of the form $P_N(t, q, p, (y - q_t)/\sqrt{\varepsilon})$ or its derivative. Those terms bring negative powers of ε . We need to estimate the worst contribution. Since P_N is polynomial in the last variables, it suffices to study the degree of P_N . We recall that

$$P_N(t, q, p, x) = \pi^{-d/4} \det(A(t, q, p) + iB(t, q, p))_c^{-1/2} \mathcal{Q}_N(t, q, p, x),$$

where \mathcal{Q}_N is given by (II.27):

$$\mathcal{Q}_N(t, q, p, x) := 1 + \sum_{(k, j) \in I_N} \varepsilon^{\frac{k}{2} - j} p_{k, j}(t, q, p, x).$$

The $p_{k, j}$ are polynomials in x of degree at most k . Evalutating $p_{k, j}$ at $(t, q, p, (y - q_t)/\sqrt{\varepsilon})$, it brings an ε -contribution of order at worst $\varepsilon^{-k/2}$ (see (II.27)). Multiplying by $\varepsilon^{k/2 - j}$ leads to a contribution of order at worse ε^{-j} in the summation. Furthermore $(k, j) \in I_N$, so we get the bounds $1 \leq j \leq 2N - 1$. So, the worst contribution of P_N is of order $\varepsilon^{-(2N-1)}$:

$$\begin{aligned} \mathcal{Q}_N(t, q, p, (y - q_t)/\sqrt{\varepsilon}) &= 1 + \sum_{(k, j) \in I_N} \varepsilon^{\frac{k}{2} - j} p_{k, j}(t, q, p, (y - q_t)/\sqrt{\varepsilon}), \\ &= \sum_{(k, j) \in I_N} O(\varepsilon^{\frac{k}{2} - j - \frac{k}{2}}) = \sum_{(k, j) \in I_N} O(\varepsilon^{-j}) = O(\varepsilon^{-(2N-1)}). \end{aligned}$$

We now compensate this loss of precision with the ε^{L+1} term. Select an integer L such that $L - 2N + 2 \geq 1$; the contribution of the whole term is

$$O\left(\varepsilon^{L+1} \sup_{k \geq 2L+d+3} \left\| \partial_{(\alpha, \beta)}^k \left(\widehat{S}(\cdot) \widehat{\phi}^*(\cdot) P_N \left(\cdot, \cdot, \cdot, \frac{\cdot}{\sqrt{\varepsilon}} \right) \chi_3 \right) \right\| \right) = O(\varepsilon^{L-2N+2}) = O(\varepsilon).$$

• **Contribution of the sum** • We still interpret the problem in the (t, X) variables (instead of (α, β)). We have to be more careful with the second integral in (II.32). If we consider

the same arguments, since ε^{L+1} is no longer here, this term should explode. Here, the key argument is that we evaluate P_N on the stationary set, *i.e.* $x = y = q_{T_R} = 0$. In particular since P_N is polynomial in x , the only contribution which remains is the coefficient of degree 0 of P_N (or the coefficient of degree 0 of its derivatives).

If the operator of derivation $Q_{2l}(\partial)$ acts only on the last variable of P_N , the only contribution is the coefficient of order 0 of $Q_{2l}(\partial)P_N$ which is a sum:

$$(\varepsilon^l Q_{2l}(\partial)P_N(\cdot, \cdot, \cdot, \cdot/\sqrt{\varepsilon}))(T_R, 0, p, 0) = \sum_{(k,j) \in I_N} \varepsilon^{\frac{k}{2}-j} Q_{2l}(\partial) p_{k,j}(T_R, 0, p, 0)$$

Since $(k, j) \in I_N$, we have $1/2 \leq \frac{k}{2} - j$. Thus, the above sum is of order $\sqrt{\varepsilon}$ and we deduce that

$$\int_{\Pi_\theta I_{\theta_0}} \exp\left(\frac{i}{\varepsilon} \psi \circ \gamma^{-1}(0, \beta)\right) \sum_{l=1}^L \varepsilon^l Q_{2l}(\partial) \left(\left(\widehat{S}(\cdot) \widehat{\phi}^*(\cdot) P_N \left(\cdot, \cdot, \cdot, \frac{\cdot}{\sqrt{\varepsilon}} \right) \right) \circ \gamma^{-1} \chi_3 \right) (0, \beta) d\beta = O(\sqrt{\varepsilon}).$$

• **Conclusion** • We just prove that $J_\varepsilon = L_\varepsilon + O_{\delta, T_1}(\sqrt{\varepsilon})$ with:

$$L_\varepsilon := \int_{\Pi_\theta I_{\theta_0}} \exp\left(\frac{i}{\varepsilon} \psi \circ \gamma^{-1}(0, \beta)\right) \left(\left(Q_0(\cdot) \widehat{S}(\cdot) \widehat{\phi}^*(\cdot) P_N \left(\cdot, \cdot, \cdot, \frac{\cdot}{\sqrt{\varepsilon}} \right) \right) \circ \gamma^{-1} \chi_3 \right) (0, \beta) d\beta.$$

Writing this integral in terms of the hyper spherical coordinates $\theta_1, \dots, \theta_{d-1}$, we have

$$L_\varepsilon = \int_{\Pi_\theta I_{\theta_0}} \frac{(2\pi)^k e^{\frac{i\pi}{4} \operatorname{sgn}\left(\frac{D^2 \psi \circ \gamma}{D\alpha^2}(0, \theta_1, \dots, \theta_{d-1})\right)}}{\det\left(\frac{D^2 \psi \circ \gamma}{D\alpha^2}(0, \theta_1, \dots, \theta_{d-1})\right)} \exp\left(\frac{i}{\varepsilon} \psi(0, 0, p, -p, 0, p, T_R)\right) \det(A(T_R, 0, p) + iB(T_R, 0, p))_c^{-1/2} \widehat{S}(p) \widehat{\phi}^*(-p) d\theta_1 \dots d\theta_{d-1},$$

The phase ψ evaluated on the set M_X is :

$$\psi(0, 0, p, -p, 0, p, T_R) = \int_0^{T_R} \left(\frac{p_s^2}{2} + n^2(q_s) \right) ds.$$

We use the symmetries to simplify. Since n^2 is radial on the cone

$$\{t.x, \text{ s.t. } t > 0 \text{ and } x \in \Pi_x M_X\},$$

we deduce that $\psi(0, 0, p, -p, 0, p, T_R) = \psi(0, 0, p_0, -p_0, 0, p_0, T_R)$ for any $p \in I_{\theta_0}$. The same holds for

$$\frac{e^{\frac{i\pi}{4} \operatorname{sgn}\left(\frac{D^2 \psi \circ \gamma}{D\alpha^2}(0, \theta_1, \dots, \theta_{d-1})\right)}}{\det\left(\frac{D^2 \psi \circ \gamma}{D\alpha^2}(0, \theta_1, \dots, \theta_{d-1})\right)} \quad \text{and} \quad \det(A(T_R, 0, p) + iB(T_R, 0, p))_c^{-1/2}.$$

Finally, L_ε can be written:

$$L_\varepsilon = C_{T_R, d} e^{\left(\frac{i}{\varepsilon} \int_0^{T_R} \left(\frac{p_s^2}{2} + n^2(q_s) \right) ds \right)} \int_{I_{\theta_0}} \widehat{S}(p) \widehat{\phi}^*(-p) d\theta_1 \dots d\theta_{d-1}, \quad (\text{II.33})$$

with

$$C_{T_R, d} := \frac{(2\pi)^{5d+2} e^{i \frac{\pi}{4} \text{sgn} \left(\frac{D^2 \psi}{D\alpha^2}(0, 0, p_0, -p_0, 0, p_0, T_R) \right)}}{\det \left(\frac{D^2 \psi}{D\alpha^2}(0, 0, p_0, -p_0, 0, p_0, T_R) \right)} \det(A(T_R, 0, p_0) + iB(T_R, 0, p_0))_c^{-1/2}.$$

This ends the proof. ■

Remark. $e^{\left(\frac{i}{\varepsilon} \int_0^{T_R} \left(\frac{p_s^2}{2} + n^2(q_s) \right) ds \right)}$ is the action associated to $h(x, \xi) = \frac{\xi^2}{2} - n^2(x)$.

Finally we can prove assertion *iii*) of Theorem [II.1](#).

Proof of Theorem [II.1](#). The proof is exactly the same as the proof performed in [\[7\]](#). According to Proposition [II.2](#) we have:

$$\frac{i}{\varepsilon} \int_0^{2T_0\varepsilon} \chi \left(\frac{t}{\varepsilon T_0} \right) \langle \exp(it(\Delta_x/2 + n^2(0))), \phi \rangle dt = \langle w_{out}, \phi \rangle + O \left(T_0^{-d/2+1} \right) + O_{T_0}(\varepsilon^0).$$

Then out of the zero-energy level according to [II.3](#),

$$\frac{1}{\varepsilon} \int_{T_0\varepsilon}^{+\infty} (1 - \chi) \left(\frac{t}{T_0\varepsilon} \right) \langle (1 - \chi_\delta(H_\varepsilon)) U_\varepsilon(t) S_\varepsilon, \phi_\varepsilon(t) \rangle dt = O \left(\frac{1}{T_0} \right) + O_{T_0}(\varepsilon^0).$$

For very large times, using [II.4](#):

$$\frac{1}{\varepsilon} \int_{T_1}^{+\infty} (1 - \chi) \left(\frac{t}{T_1} \right) e^{-\alpha_\varepsilon t} \langle \chi_\delta(H_\varepsilon) U_\varepsilon(t) S_\varepsilon, \phi_\varepsilon \rangle dt = O_{\delta, T_1}(\varepsilon^0).$$

Using [\(II.16\)](#),

$$\frac{1}{\varepsilon} \int_{T_0\varepsilon}^\theta \chi \left(\frac{t}{T_0} \right) \left(1 - \chi \left(\frac{t}{T_0\varepsilon} \right) \right) e^{-\alpha_\varepsilon t} \langle U_\varepsilon(t) \chi_\delta(H_\varepsilon) S_\varepsilon, \phi_\varepsilon \rangle dt = O_\delta \left(T_0^{-d/2+1} \right).$$

Finally unlike [II.3](#), the contribution of

$$\frac{1}{\varepsilon} \int_\theta^{T_1} \chi \left(\frac{t}{T_0} \right) \left(1 - \chi \left(\frac{t}{T_0\varepsilon} \right) \right) e^{-\alpha_\varepsilon t} \langle U_\varepsilon(t) \chi_\delta(H_\varepsilon) S_\varepsilon, \phi_\varepsilon \rangle dt$$

is not vanishing. We have explicitly computed its contribution assuming S radiates towards the bump of potential:

$$\frac{1}{\varepsilon} \int_\theta^{T_1} \chi \left(\frac{t}{T_0} \right) \left(1 - \chi \left(\frac{t}{T_0\varepsilon} \right) \right) e^{-\alpha_\varepsilon t} \langle U_\varepsilon(t) \chi_\delta(H_\varepsilon) S_\varepsilon, \phi_\varepsilon \rangle dt = L_\varepsilon + O(\sqrt{\varepsilon}).$$

Choosing conveniently the parameters δ , T_0 and T_1 , the proposition is proved. ■

Remark. The hypothesis on the support of \widehat{S} is a technical hypothesis. It allows us not to deal with the edge of the stationary set.

Nevertheless, the contribution on the edge can be taken into account if more information is known on the linearized flow of order k . If the linearized flow of order k is not vanishing on the orthonormal direction of $T_m M$ at time T_R , then we can apply the stationary phase theorem (with an edge). In that case the oscillating integral can still be computed. The limit is the same as described in this case.

4 Appendix

In this section M is supposed to be a submanifold.

In this paper we have constructed a potential such that the stationary set has an edge. In this appendix we discuss the topology of the stationary set when it is a submanifold. It confirms that the way we look at the potential n^2 has to lead a stationary set with an edge. Then we give some hints to construct a potential such that the set M is a submanifold of dimension $d - 1$.

Proposition II.17. *Let n^2 be a non-trapping potential. Then*

i) M is a compact submanifold of $\mathbb{R}^{2d} \times]0, +\infty[$,

ii) M has a finite number of connex components.

Proof of proposition II.17 We only prove assertion *i*). Assertion *ii*) is a direct consequence of *i*). Since the dimension is finite we just prove that M is closed and bounded.

◦ M is a closed set as the pull-back of a closed set by a smooth function.

◦ M is bounded in the directions ξ and η . There remains to check that M is bounded in the direction t . n^2 is a non-trapping potential. Any trajectory issued from 0 goes back to 0 a finite number of times. This number depends (*a priori*) on the initial speed $\xi \in \mathbb{S}^{d-1}$.

Nevertheless, the solution of Hamilton equation is continuous relatively to the initial data. Thus an argument of compacity allows to conclude that any trajectory issued from 0 goes back to 0 a finite number of times. This number is bounded independently of ξ . ■

Remark. When the potential is not non-trapping, everything can happen.

Let us examine what can happen in the limiting case *i.e.* $\dim M = d - 1$:

Proposition II.18. *Suppose M to be a submanifold of dimension $d - 1$. Consider the finite family $\{U_j\}_{j \in \{1, \dots, N\}}$ of the connex component of M . The projection on space time variable of U_j , denoted by $\Pi_t U_j$, is an interval with a non-empty interior.*

Proof of Proposition II.18. We proceed by contradiction. We assume that for any $j \in \{1, \dots, N\}$, the projection $\Pi_t U_j$ is reduced to a point t_j . We denote by $\Pi_\xi U_j$ and $\Pi_\eta U_j$ the projections of U_j on the space variables ξ and η respectively.

◦ We prove that U_j is diffeomorphic to a submanifold of \mathbb{S}^{d-1} . To do so we exhib the diffeomorphism. For all $j \in \{0, \dots, N\}$, consider the application F_j defined by :

$$\begin{cases} F_j : & U_j & \longrightarrow & \Pi_\xi U_j \subset \mathbb{S}^{d-1}, \\ & (\xi, \eta, t_j) & \longmapsto & \xi. \end{cases}$$

F_j is invertible. Indeed, consider g_j the application $g_j : \Pi_\xi U_j \longrightarrow \Pi_\eta U_j$ which associates to ξ the value $g(\xi) := \Xi(t_j, 0, \xi)$. We construct the inverse of F_j

$$\begin{cases} F_j : \Pi_\xi U_j \subset \mathbb{S}^{d-1} & \longrightarrow & U_j, \\ & \xi & \longmapsto & (\xi, g(\xi), t_j). \end{cases}$$

F_j and G_j are smooth, thus U_j can be seen as a compact submanifold of \mathbb{S}^{d-1} of dimension $d - 1$.

◦ In other words for any $j \in \{1, \dots, N\}$, U_j is homeomorphic to \mathbb{S}^{d-1} . It means that any trajectory with initial data $(0, \xi) \in H_0$ goes back to 0 in a finite time t_j . First it implies that $t_j = t_i$. Then the trajectories are periodic of period t_j . This is absurd because the trajectories are assumed not to be trapped. ■

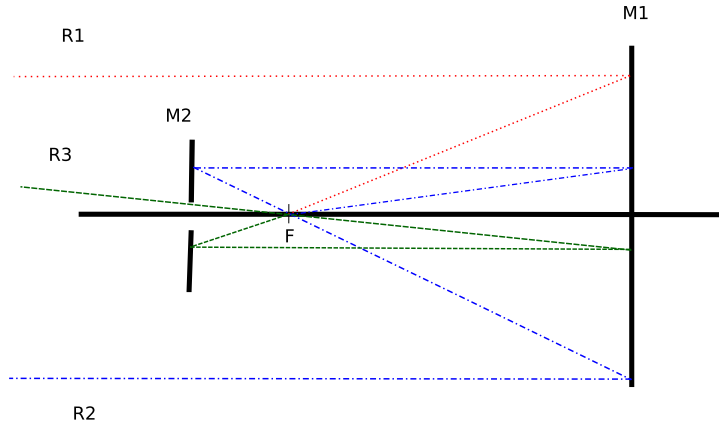


Figure II.7: Trajectories are not trapped with two parabolic mirrors.

Due to the difficulty of the computation we did not exhibit a counterexample such that $\Pi_t U_i$ is an interval with a non-empty interior.

Let us give some clues to build such a potential. Consider two parabolic mirrors with the same focal axis and focal point (F). Furthermore we suppose that the focal distances are different. The trajectories issued from 0 go back to 0 in a different time. The potential is not non-trapping. It suffices to perforate the smallest mirror at the level of the focal axis.

Depending on the angle two cases can occur :

- the trajectory leaves by passing up or down the mirrors $M2$. (see rays $R1$ and $R2$ on Figure II.7),
- the trajectory tends to be closer to the focal axis. Then, the trajectory leaves by the hole in the mirror $M2$ (see the ray $R3$ on Figure II.7).

Chapitre III

Production d'une dissipation par interactions d'ondes oscillantes dans une équation de Navier-Stokes forcée

Abstract : We consider a bidimensionnal Navier-Stokes type equation. Typical wavelength of the oscillations considered here is $1/\varepsilon$. We force one variable to oscillate like $1/\varepsilon^2$ thanks to a polarized source term. We study the interactions between those oscillations.

To be more accurate, we consider a family of exact solutions that we perturb at initial time $t = 0$. We prove that the oscillating Cauchy problem associated with this new initial data is well-posed. To do so we exhibit a complete expansion of the solution as ε goes to 0. This expansion reveals a boundary layer in time for the velocity. A noticeable aspect is the creation of some dissipation on the mean term of the velocity due to some *drift-diffusion* mechanism.

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1 Introduction

In Section 1, we introduce the underlying equations and the functional framework. Then, we state our main result.

1.1 The equations

The time and space variables are $t \in \mathbb{R}_+$ and $x := (x_1, x_2) \in \mathbb{R}^2$. The state variables are the density $\rho \in \mathbb{R}_+$ and the two components u_1 and u_2 of the velocity of the fluid $u := {}^t(u_1, u_2) \in \mathbb{R}^2$. Given a function $u : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, note as usual:

$$\operatorname{div} u := \partial_1 u_1 + \partial_2 u_2, \quad \partial_1 := \frac{\partial}{\partial x_1}, \quad \partial_2 := \frac{\partial}{\partial x_2}.$$

In what follows, $\varepsilon \in]0, 1]$ is a parameter approaching zero. Introduce the dissipation:

$$\mathcal{P}_{\varepsilon, \lambda_\varepsilon} u = {}^t(\mathcal{P}_{\varepsilon, \lambda_\varepsilon}^1 u, \mathcal{P}_{\varepsilon, \lambda_\varepsilon}^2 u) := \mu \varepsilon^2 \Delta_x u + \lambda_\varepsilon \varepsilon^2 \nabla \operatorname{div} u$$

where $\mu \in \mathbb{R}_+^*$ is fixed whereas $\lambda_\varepsilon \in \mathbb{R}_+^*$ goes to 0 as ε goes to 0. Let h be some smooth periodic function with mean zero:

$$h : \mathbb{T} \rightarrow \mathbb{R}, \quad \mathbb{T} := \mathbb{R}/\mathbb{Z}, \quad h \in \mathcal{C}^\infty(\mathbb{T}; \mathbb{R}), \quad \int_{\mathbb{T}} h(\theta) d\theta = 0.$$

Consider the following oscillation which is polarized on the second component:

$$F_\varepsilon(x) = {}^t(0, F_\varepsilon^2)(x) := \varepsilon^{-2} {}^t(0, \mu \partial_{\theta\theta}^2 h(\varepsilon^{-2} x_1)), \quad \varepsilon \in]0, 1].$$

Our starting point is the study of a model based on two-dimensional compressible isentropic equations of the Navier-Stokes type, as can be found in [6, 28], forced by the source term F_ε :

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla \rho^\gamma = \rho(\mathcal{P}_{\varepsilon, \lambda_\varepsilon} u - F_\varepsilon), \end{cases} \quad u \otimes u := \begin{pmatrix} u_1 u_1 & u_1 u_2 \\ u_1 u_2 & u_2 u_2 \end{pmatrix}.$$

To obtain a quasilinear system having a symmetric form, it is classical [30] to introduce the state variable $p := \frac{\sqrt{\gamma}}{C} \rho^C$ with $C := \frac{\gamma-1}{2}$. Then, we have to deal with:

$$\begin{cases} \partial_t p + u \cdot \nabla p + C p \operatorname{div} u = 0, \\ \partial_t u + u \cdot \nabla u + C p \nabla p = \mathcal{P}_{\varepsilon, \lambda_\varepsilon} u - F_\varepsilon. \end{cases} \quad (\text{III.1})$$

Observe that:

$$\mathcal{P}_{\varepsilon, \lambda_\varepsilon} {}^t(0, h_\varepsilon) - F_\varepsilon = 0, \quad h_\varepsilon(x) := h\left(\frac{x_1}{\varepsilon^2}\right), \quad \forall \varepsilon \in]0, 1].$$

It follows that, for all $\varepsilon \in]0, 1]$, the oscillation ${}^t(0, 0, h_\varepsilon)$ satisfies Equation (III.1). Our aim is to consider the problem of the *stability* of such families of solutions. To this end, at the initial time $t = 0$, we modify ${}^t(0, 0, h_\varepsilon)$ by adding some perturbation. More precisely, we start with:

$$(p, u^1, u^2)(0, x) = (0, 0, h)\left(\frac{x_1}{\varepsilon^2}\right) + (\varepsilon^\nu q_{0, \varepsilon}, \varepsilon^M v_{0, \varepsilon}^1, \varepsilon^M v_{0, \varepsilon}^2)\left(\frac{x_1}{\varepsilon^2}, \frac{x_2}{\varepsilon}\right) \quad (\text{III.2})$$

where $(\nu, M) \in \mathbb{N}^2$ with ν large enough and $M \geq 7/2$ (retain that $\nu \gg M$), whereas:

$$(q_{0,\varepsilon}, v_{0,\varepsilon}^1, v_{0,\varepsilon}^2)(\theta, y) \in H^\infty(\mathbb{T} \times \mathbb{R}; \mathbb{R}^3), \quad y := \frac{x_2}{\varepsilon} \in \mathbb{R}.$$

One effect of the above perturbation is to introduce a dependence on $x_2 \in \mathbb{R}$ (or $y \in \mathbb{R}$). Despite the smallness of ε^ν (and maybe ε^M), when solving (III.1)-(III.2), we have to understand the interactions that occur between the very fast oscillations in the direction x_1 (with wavelength ε^2) and the fast variations in the transversal direction x_2 (with wavelength ε). On this way, we are faced with questions about *turbulence*, in the spirit of models proposed in [9, 10, 12].

Another insight on the subject can be obtained by looking at (III.1) in the variables $(\theta, y) \in \mathbb{T} \times \mathbb{R}$. Then, we are faced with a hyperbolic-parabolic system implying some singular (in $\varepsilon \in]0, 1]$) symmetric quasilinear part:

$$\begin{cases} \partial_t p + \varepsilon^{-2} (u^1 \partial_\theta p + \varepsilon u^2 \partial_y p) + C \varepsilon^{-2} p (\partial_\theta u^1 + \varepsilon \partial_y u^2) = 0, \\ \partial_t u^1 + \varepsilon^{-2} (u^1 \partial_\theta u^1 + \varepsilon u^2 \partial_y u^1) + C \varepsilon^{-2} p \partial_\theta p = \tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon}^1 u, \\ \partial_t u^2 + \varepsilon^{-2} (u^1 \partial_\theta u^2 + \varepsilon u^2 \partial_y u^2) + C \varepsilon^{-1} p \partial_y p = \tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon}^2 u - F_\varepsilon^2, \end{cases} \quad (\text{III.3})$$

and some viscosity which is degenerated on the density and becomes large when $\varepsilon \rightarrow 0$:

$$\tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon} := \begin{pmatrix} \tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon}^1 \\ \tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon}^2 \end{pmatrix} = \frac{1}{\varepsilon^2} \begin{pmatrix} \mu (\partial_{\theta\theta} u^1 + \varepsilon^2 \partial_{yy} u^1) + \lambda_\varepsilon (\partial_{\theta\theta} u^1 + \varepsilon \partial_{\theta y} u^2) \\ \mu (\partial_{\theta\theta} u^2 + \varepsilon^2 \partial_{yy} u^2) + \lambda_\varepsilon (\varepsilon \partial_{\theta y} u^1 + \varepsilon^2 \partial_{yy} u^2) \end{pmatrix}.$$

In this chapter, we show that (for ν large enough and $M \geq 7/2$) the *oscillating Cauchy problem* (III.1)-(III.2) is locally well posed in time. We prove (Theorem III.4) the existence of a time $T \in \mathbb{R}_+^*$ independent of $\varepsilon \in]0, 1]$ with solutions $(p_\varepsilon, u_\varepsilon^1, u_\varepsilon^2) = (\varepsilon^\nu q_\varepsilon, \varepsilon^M v_\varepsilon^1, h_\varepsilon + \varepsilon^M v_\varepsilon^2)$ of (III.1)-(III.2) on the interval $[0, T]$. We also exhibit (Propositions III.1 and III.2) a complete expansion as ε approaches 0 for the expression $(q_\varepsilon, v_\varepsilon^1, v_\varepsilon^2)$. We find (in a sense to be precised later) that $q_\varepsilon \simeq q_\varepsilon^a$ and $v_\varepsilon := (v_\varepsilon^1, v_\varepsilon^2) \simeq v_\varepsilon^a := (v_\varepsilon^{a1}, v_\varepsilon^{a2})$ with:

$$q_\varepsilon^a(t, \theta, y) = \sum_{k=0}^{N+1} \varepsilon^k q_k^\varepsilon(t, \theta, y), \quad v_\varepsilon^a(t, \theta, y) = \sum_{k=0}^{N+1} \varepsilon^k \left(v_k^s(t, \theta, y) + v_k^f\left(\frac{t}{\varepsilon^2}, \theta, y\right) \right). \quad (\text{III.4})$$

These expansions reveal some time boundary layer at time $t = 0$ (recorded at the level of the contribution $v_k^f(\tau, \cdot)$ which is exponentially decreasing with respect to the variable τ) together with some mean evolution behaviour (described by v_k^s). A noticeable aspect is the production of some dissipation when looking at the transport equation (III.9) on v_k^s . The present approach is not in the continuation of usual $k - \varepsilon$ models [31]. But, in the same spirit, it confirms (and justifies) that the interaction of oscillations can indeed be described at a macroscopic level by the introduction of some *turbulent viscosity*.

1.2 The functional framework

1.2.1 Sobolev spaces

Here K denotes \mathbb{R} , $\mathbb{T} \times \mathbb{R}$ or \mathbb{R}^2 . Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$. The length of α is $|\alpha| := \alpha_1 + \alpha_2$. The notation ∂^α is for the differential operator $\partial_\theta^{\alpha_1} \partial_y^{\alpha_2}$.

- Given $m \in \mathbb{N} \cup \{+\infty\}$ and $p \in \mathbb{N}^* \cup \{+\infty\}$, recall that $W^{m,p}$ is:

$$W^{m,p} := \{f \in L^p(K) ; \partial^\alpha f \in L^p(K), |\alpha| \leq m\}, \quad H^m := W^{m,2}.$$

When $m \in \mathbb{N}$, the space $W^{m,p}$ can be equipped with the following semi-norm and norm:

$$\forall p \in \mathbb{N}^* \cup \{+\infty\}, \quad \|f\|_{W^{m,p}}^\circ := \sum_{\alpha \in \mathbb{N}^2, |\alpha|=m} \|\partial^\alpha f\|_{L^p}, \quad \|f\|_{W^{m,p}} := \sum_{k=0}^m \|f\|_{W^{k,p}}^\circ.$$

For $s \in \mathbb{R}_+ \setminus \mathbb{N}$, we can still define spaces $W^{s,p}$ and H^s by interpolation theory.

- Let $(m, n) \in \mathbb{N}^2$ with $n \leq m$. Define the functional spaces:

$$\mathcal{W}_T^{m,n} := \left\{ f ; f \in C^j([0, T]; W^{m-j, \infty}), \forall j \in \{0, \dots, n\} \right\},$$

$$\mathcal{H}_T^{m,n} := \left\{ f ; f \in C^j([0, T]; H^{m-j}), \forall j \in \{0, \dots, n\} \right\}, \quad T \in \mathbb{R}_+ \cup \{+\infty\},$$

which can be seen as Banach spaces when provided with the norms:

$$\|f\|_{\mathcal{W}_T^{m,n}} := \sup_{t \in [0, T]} \sum_{j=0}^n \|\partial_t^j f(t, \cdot)\|_{W^{m-j, \infty}}, \quad \|f\|_{\mathcal{H}_T^{m,n}} := \sup_{t \in [0, T]} \sum_{j=0}^n \|\partial_t^j f(t, \cdot)\|_{H^{m-j}}.$$

- In order to deal with functions $f(t, \cdot)$ defined on $\mathbb{R}_+ \times K$, which are exponentially decreasing in the time $t \in \mathbb{R}_+$, and which take their values in the Sobolev space H^s , define:

$$\mathcal{E}_\delta^s := \left\{ f ; \sup_{t \in [0, +\infty[} (e^{\delta t} \|f(t, \cdot)\|_{H^s(K)}) < +\infty \right\}, \quad \delta \in \mathbb{R}_+^*.$$

- Finally, introduce $\mathcal{E}_\delta^\infty := \bigcap_{j \in \mathbb{N}} \mathcal{E}_\delta^j$, $\mathcal{V}_T^{\infty, 0} := \bigcap_{j \in \mathbb{N}} \mathcal{V}_T^{j, 0}$ and $\mathcal{V}_T^\infty := \bigcap_{j \in \mathbb{N}} \mathcal{V}_T^{j, j}$ where $\mathcal{V} \in \{\mathcal{H}, \mathcal{W}\}$.

1.2.2 Families of functions

In this paragraph, we fix some $\varepsilon_0 \in]0, 1]$ and look at families of the type $\{f_\varepsilon\}_{\varepsilon \in]0, \varepsilon_0]}$.

- Assume that $f_\varepsilon \in W^{m,p}(K)$ for all $\varepsilon \in]0, \varepsilon_0]$. To control the size of f_ε , we can use the following weighted anisotropic semi-norm and norm:

$$\forall p \in \mathbb{N}^* \cup \{+\infty\}, \quad \|f_\varepsilon\|_{W_{(1,\varepsilon)}^{m,p}}^\circ := \sum_{\alpha \in \mathbb{N}^2, |\alpha|=m} \|\varepsilon^{\alpha_1} \partial^\alpha f_\varepsilon\|_{L^p}, \quad \|f_\varepsilon\|_{W_{(1,\varepsilon)}^{m,p}} := \sum_{k=0}^m \|f_\varepsilon\|_{W_{(1,\varepsilon)}^{k,p}}^\circ.$$

We will say that $\{f_\varepsilon\}_\varepsilon$ is bounded in $W_{(1,\varepsilon)}^{m,p}$ when:

$$\|f_\cdot\|_{W_{(1,\cdot)}^{m,p}} := \sup_{\varepsilon \in]0, \varepsilon_0]} \|f_\varepsilon\|_{W_{(1,\varepsilon)}^{m,p}} < +\infty.$$

- Assume that $f_\varepsilon \in \mathcal{V}_T^{m,n}$ for all $\varepsilon \in]0, \varepsilon_0]$ where $\mathcal{V} = \mathcal{W}$ or $\mathcal{V} = \mathcal{H}$. To control the size of f_ε , we can use the following norms:

$$\|f_\varepsilon\|_{\mathcal{V}_{T,\varepsilon}^{m,n}} := \sup_{t \in [0, T]} \sum_{j=0}^{\lfloor n/2 \rfloor} \|\varepsilon^{2j} \partial_t^j f_\varepsilon(t, \cdot)\|_{V^{m-j}}, \quad \|f_\varepsilon\|_{\mathcal{V}_{T,(1,\varepsilon)}^{m,n}} := \sup_{t \in [0, T]} \sum_{j=0}^{\lfloor n/2 \rfloor} \|\varepsilon^{2j} \partial_t^j f_\varepsilon(t, \cdot)\|_{V_{(1,\varepsilon)}^{m-j}}.$$

We will say that $\{f_\varepsilon\}_\varepsilon$ is bounded in $\mathcal{V}_{T,\varepsilon}^{m,n}$ or in $\mathcal{V}_{T,(1,\varepsilon)}^{m,n}$ when we have respectively:

$$\|f\|_{\mathcal{V}_{T,\cdot}^{m,n}} := \sup_{\varepsilon \in]0, \varepsilon_0]} \|f_\varepsilon\|_{\mathcal{V}_{T,\varepsilon}^{m,n}} < +\infty, \quad \|f\|_{\mathcal{V}_{T,(1,\cdot)}^{m,n}} := \sup_{\varepsilon \in]0, \varepsilon_0]} \|f_\varepsilon\|_{\mathcal{V}_{T,(1,\varepsilon)}^{m,n}} < +\infty.$$

Classical embedding : for $s > 1$ we have $H^s(\mathbb{T} \times \mathbb{R}) \hookrightarrow W^{0,\infty}(\mathbb{T} \times \mathbb{R}) \equiv L^\infty(\mathbb{T} \times \mathbb{R})$. When taking into account the dependence on $\varepsilon \in]0, 1]$, there is a loss of powers in ε . Retain here that:

$$\exists C \in \mathbb{R}_*^+; \quad \|f_\varepsilon\|_{L^\infty(\mathbb{T} \times \mathbb{R})} \leq C \varepsilon^{-1/2} \|f_\varepsilon\|_{H_{(1,\varepsilon)}^s(\mathbb{T} \times \mathbb{R})}, \quad \forall \varepsilon \in]0, \varepsilon_0]. \quad (\text{III.5})$$

1.2.3 Decomposition of a periodic function

Any function $u \in L^2(\mathbb{T}; \mathbb{R})$ can be decomposed as:

$$u(\theta) = \langle u \rangle + u^*(\theta), \quad \langle u \rangle \in \mathbb{R}, \quad u^* \in L^2(\mathbb{T}; \mathbb{R}), \quad \langle u \rangle \equiv \Pi u := \int_{\mathbb{T}} u(\theta) d\theta.$$

According to this decomposition, we clearly have that $\Pi u^* \equiv 0$.

When the function u depends on extra variables than θ , like (t, y) , we can still perform the preceding operations. For instance:

$$u(t, \theta, y) = \langle u \rangle(t, y) + u^*(t, \theta, y), \quad \Pi u(t, y) := \int_{\mathbb{T}} u(t, \theta, y) d\theta.$$

The derivative ∂_θ acts in the sense of distributions on the space $L^2(\mathbb{T}; \mathbb{R})$. We find:

$$\mathcal{K} := \ker \partial_\theta = \{u \equiv c; c \in \mathbb{R}\}, \quad \mathcal{K}^\perp := (\ker \partial_\theta)^\perp = \{u \in L^2(\mathbb{T}; \mathbb{R}); \Pi u = 0\}.$$

The action ∂_θ has a (right) inverse $\partial_\theta^{-1} : \mathcal{K}^\perp \longrightarrow \mathcal{K}^\perp \cap H^1(\mathbb{T}; \mathbb{R})$ which is given by:

$$\partial_\theta^{-1} u(\theta) := \int_0^\theta u(s) ds - \int_{\mathbb{T}} \int_0^\theta u(s) ds d\theta, \quad \forall \theta \in \mathbb{T}.$$

Given a functional space $V \subset L^2(\mathbb{T}; \mathbb{R})$, we adopt the convention $V^\perp := V \cap \mathcal{K}^\perp$.

1.3 Main statements

Since we impose $\nu \gg M \geq 7/2$, the equations on the components u^1 and u^2 can be considered as being partially decoupled from the equation on p . Up to some extent, we can first deal with:

$$\begin{aligned} \mathcal{L}_1(\varepsilon, q_\varepsilon, v_\varepsilon) &:= \partial_t v_\varepsilon^1 + \varepsilon^{-1} h \partial_y v_\varepsilon^1 + \varepsilon^{M-2} (v_\varepsilon^1 \partial_\theta v_\varepsilon^1 + \varepsilon v_\varepsilon^2 \partial_y v_\varepsilon^1) + C \varepsilon^{2\nu-M-2} q_\varepsilon \partial_\theta q_\varepsilon - \tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon}^1 v_\varepsilon, \\ \mathcal{L}_2(\varepsilon, q_\varepsilon, v_\varepsilon) &:= \partial_t v_\varepsilon^2 + \varepsilon^{-1} h \partial_y v_\varepsilon^2 + \varepsilon^{-2} \partial_\theta h v_\varepsilon^1 + \varepsilon^{M-2} (v_\varepsilon^1 \partial_\theta v_\varepsilon^2 + \varepsilon v_\varepsilon^2 \partial_y v_\varepsilon^2) \\ &\quad + C \varepsilon^{2\nu-M-1} q_\varepsilon \partial_y q_\varepsilon - \tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon}^2 v_\varepsilon, \end{aligned}$$

and then look at the remaining part as a transport equation on q_ε :

$$\mathcal{L}_0(\varepsilon, q_\varepsilon, v_\varepsilon) := \partial_t q_\varepsilon + \varepsilon^{-1} h \partial_y q_\varepsilon + \varepsilon^{M-2} (v_\varepsilon^1 \partial_\theta q_\varepsilon + \varepsilon v_\varepsilon^2 \partial_y q_\varepsilon) + C \varepsilon^{M-2} q_\varepsilon (\partial_\theta v_\varepsilon^1 + \varepsilon \partial_y v_\varepsilon^2).$$

In what follows, the results will be expressed in terms of the quantities q_ε and v_ε . Of course, the expression $(p_\varepsilon, u_\varepsilon^1, u_\varepsilon^2) = (\varepsilon^\nu q_\varepsilon, \varepsilon^M v_\varepsilon^1, h + \varepsilon^M v_\varepsilon^2)$ is a solution of (III.3) if and only if:

$$\mathcal{L}_j(\varepsilon, q_\varepsilon, v_\varepsilon) = 0, \quad \forall j \in \{0, 1, 2\}. \quad (\text{III.6})$$

1.3.1 Construction of approximated solutions

We start by constructing approximated solutions for the system (III.6). The first step is to look at the two last equations of (III.6), where the $O(\varepsilon^{2\nu-M-2}) \ll 1$ contributions (implying q_ε) are neglected. Thus, we start by considering the system:

$$\begin{cases} \mathcal{L}_1^a(\varepsilon, v_\varepsilon) := \partial_t v_\varepsilon^1 + \varepsilon^{-1} h \partial_y v_\varepsilon^1 + \varepsilon^{M-2} (v_\varepsilon^1 \partial_\theta v_\varepsilon^1 + \varepsilon v_\varepsilon^2 \partial_y v_\varepsilon^1) - \tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon}^1 v_\varepsilon, \\ \mathcal{L}_2^a(\varepsilon, v_\varepsilon) := \partial_t v_\varepsilon^2 + \varepsilon^{-1} h \partial_y v_\varepsilon^2 + \varepsilon^{-2} \partial_\theta h v_\varepsilon^1 + \varepsilon^{M-2} (v_\varepsilon^1 \partial_\theta v_\varepsilon^2 + \varepsilon v_\varepsilon^2 \partial_y v_\varepsilon^2) - \tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon}^2 v_\varepsilon. \end{cases}$$

Proposition III.1. *Assume $\lambda_\varepsilon = \lambda \varepsilon$. Fix an integer $M \in \mathbb{N}$ with $M \geq 2$. Choose any integer $N \in \mathbb{N}$ and any decay rate $\delta \in]0, \mu[$. Select any functions $v_k^0 \in H^\infty(\mathbb{T} \times \mathbb{R}; \mathbb{R}^2)$ indexed by $k \in \{0, \dots, N+1\}$. There are functions*

$$v_k^s \in \bigcap_{T \in \mathbb{R}_+^*} \mathcal{H}_T^\infty, \quad v_k^f \in \mathcal{E}_\delta^\infty, \quad k \in \{0, \dots, N+1\}$$

such that the family $\{v_\varepsilon^a\}_\varepsilon$ defined as indicated in (III.4) satisfies the following conditions:

i) At the initial time $t = 0$, the trace $v_\varepsilon^a(0, \cdot)$ is prescribed in the following way:

$$v_\varepsilon^a(0, \theta, y) = \sum_{k=0}^{N+1} \varepsilon^k v_k^0(\theta, y). \quad (\text{III.7})$$

ii) For all time $T \in \mathbb{R}_+^*$, for all $m \in \mathbb{N}$, the family $\{\varepsilon^{-N} \mathcal{L}_j^a(\varepsilon, v_\varepsilon^a)\}_\varepsilon$ with $j = 1$ or $j = 2$ is bounded in $\mathcal{H}_{T, \varepsilon}^{m, 0}$ in the sense that:

$$\sup_{\varepsilon \in]0, 1]} \sup_{t \in [0, T]} \max_{j \in \{1, 2\}} \|\varepsilon^{-N} \mathcal{L}_j^a(\varepsilon, v_\varepsilon^a)\|_{H^m(\mathbb{T} \times \mathbb{R})} < +\infty. \quad (\text{III.8})$$

Furthermore, the expression Πv_k^s is determined through an equation of the form:

$$\partial_t \Pi v_k^s - \left(\mu + \frac{1}{\mu} \Pi((\partial_\theta^{-1} h)^2) \right) \partial_{yy} \Pi v_k^s = S_k \quad (\text{III.9})$$

where the source term S_k depends only on the v_j^s with $j \leq k-1$.

To complete v_ε^a into some approximated solution $(q_\varepsilon^a, v_\varepsilon^a)$ of the complete system (III.3), there remains to identify the pressure component q_ε^a . To this end, we are satisfied to solve directly the transport equation $\mathcal{L}_0(\varepsilon, q_\varepsilon^a, v_\varepsilon^a) = 0$ where v_ε^a is adjusted as in Proposition III.1. Since the expression v_ε^a is a function of the scales of time t and $\frac{t}{\varepsilon^2}$, that goes for $q_\varepsilon^a(t, y, \theta)$ too. In what follows, we will not need to precise the way by which q_ε^a depends on the different time scales $(t, \frac{t}{\varepsilon}, \frac{t}{\varepsilon^2}, \dots)$.

Proposition III.2. *The context is as in Proposition III.1. Note $\{v_\varepsilon^a\}_\varepsilon$ the family issued from the Proposition III.1. Select functions $q_k^0 \in H^\infty(\mathbb{T} \times \mathbb{R}; \mathbb{R}^3)$ indexed by $k \in \{0, \dots, N+1\}$. There are functions q_k^ε with:*

$$\{q_k^\varepsilon\}_\varepsilon \in \bigcap_{T \in \mathbb{R}_+^*} \mathcal{H}_T^{\infty, 0}, \quad k \in \{0, \dots, N+1\}, \quad \varepsilon \in]0, 1]$$

such that the expressions q_ε^a defined as indicated in (III.4) are solutions of the Cauchy problem:

$$\mathcal{L}_0(\varepsilon, q_\varepsilon^a, v_\varepsilon^a) = 0, \quad q_\varepsilon^a(0, \theta, y) = \sum_{k=0}^{N+1} \varepsilon^k q_k^0(\theta, y). \quad (\text{III.10})$$

Moreover, for all time $T \in \mathbb{R}_+^*$, for all $m \in \mathbb{N}$ and for all $k \in \{0, \dots, N+1\}$, the family $\{q_k^\varepsilon\}_\varepsilon$ is bounded in $\mathcal{H}_{T, (1, \varepsilon)}^{m, 0}$ in the sense that:

$$\sup_{\varepsilon \in]0, 1]} \sup_{t \in [0, T]} \|q_k^\varepsilon(t, \cdot)\|_{H_{(1, \varepsilon)}^m(\mathbb{T} \times \mathbb{R})} < +\infty. \quad (\text{III.11})$$

Coming back to \mathcal{L}_1 and \mathcal{L}_2 , we can now make the following statement.

Proposition III.3. *Select $m, M, N, \nu \in \mathbb{N}$ satisfying:*

$$M \geq 2, \quad m \geq 2 \quad \text{and} \quad 2\nu - M - 5/2 - (m+1) - N \geq 0. \quad (\text{III.12})$$

Note $\{v_\varepsilon^a\}_\varepsilon$ and $\{q_\varepsilon^a\}_\varepsilon$ the families obtained with Propositions III.1 and III.2. Then, for all $j \in \{1, 2\}$, we have:

$$\sup_{\varepsilon \in]0, 1]} \sup_{t \in [0, T]} \|\varepsilon^{-N} \mathcal{L}_j(\varepsilon, q_\varepsilon^a, v_\varepsilon^a)\|_{H^m(\mathbb{T} \times \mathbb{R})} < +\infty.$$

1.3.2 Existence and stability result

The parameter $\varepsilon \in]0, 1]$ being fixed, the local in time well-posedness of the Cauchy problem (III.1)-(III.2) is standard, with corresponding solutions $(p_\varepsilon^e, u_\varepsilon^e) = (\varepsilon^\nu q_\varepsilon^e, \varepsilon^M v_\varepsilon^{e1}, h + \varepsilon^M v_\varepsilon^{e2})$. It means that, for all $\varepsilon \in]0, 1]$, there is a time $T_\varepsilon \in \mathbb{R}_+^*$ (eventually shrinking to zero when ε goes to zero) such that $(q_\varepsilon^e, v_\varepsilon^e)$ with $v_\varepsilon^e := (v_\varepsilon^{e1}, v_\varepsilon^{e2})$ is a solution of (III.6) on the time interval $[0, T_\varepsilon]$ with initial data as indicated at the level of (III.7) and (III.10).

Fix any $R \in \mathbb{N}$. We can always define on the strip $[0, T_\varepsilon]$, two functions q_ε^R and v_ε^R through the identity $(q_\varepsilon^e, v_\varepsilon^e) = (q_\varepsilon^a, v_\varepsilon^a) + \varepsilon^R (q_\varepsilon^R, v_\varepsilon^R)$. Two questions are solved below: the existence of exact solutions of (III.1)-(III.2) on a time interval $[0, T_c]$ with $T_c \in \mathbb{R}_+^*$ independent of $\varepsilon \in]0, 1]$ and the production of controls on $(q_\varepsilon^R, v_\varepsilon^R)$ showing that $(q_\varepsilon^a, v_\varepsilon^a)$ gives indeed some good asymptotic description of $(q_\varepsilon^e, v_\varepsilon^e)$ on $[0, T_c]$.

Theorem III.4. *Assume $\lambda_\varepsilon := \lambda \varepsilon$ with $\lambda < 4\mu$. Let $m, \nu, M, N, R \in \mathbb{N}$ satisfying $M \geq 7/2$ and*

$$w_m := \min(2\nu - M - 5/2 - (m+3) - R, N - R) \geq 0. \quad (\text{III.13})$$

Then the following statements hold.

i) *There exist $T_c > 0$ and $\varepsilon_{crit} > 0$ such that*

$$\forall \varepsilon \in]0, \varepsilon_{crit}], \quad T_\varepsilon \geq T_c. \quad (\text{III.14})$$

ii) *The family $\{q_\varepsilon^R\}_\varepsilon$ is bounded in $\mathcal{H}_{T_c, (1, \varepsilon_c)}^{m+3, 0}$:*

$$\sup_{\varepsilon \in]0, \varepsilon_c]} \sup_{t \in [0, T_c]} \|q_\varepsilon^R(t, \cdot)\|_{H_{(1, \varepsilon)}^{m+3}} < +\infty. \quad (\text{III.15})$$

iii) The family $\{v_\varepsilon^{1R}\}_\varepsilon$ is bounded in $\mathcal{H}_{T_c, \varepsilon_c}^{m+3,0}$:

$$\sup_{\varepsilon \in]0, \varepsilon_c]} \sup_{t \in [0, T_c]} \|v_\varepsilon^{1R}(t, \cdot)\|_{H^{m+3}} < +\infty. \quad (\text{III.16})$$

iv) The family $\{v_\varepsilon^{2R}\}_\varepsilon$ is bounded in $\mathcal{H}_{T_c, \varepsilon_c}^{m+3,0}$:

$$\sup_{\varepsilon \in]0, \varepsilon_c]} \sup_{t \in [0, T_c]} \|\varepsilon v_\varepsilon^{2R}(t, \cdot)\|_{H^{m+3}} < +\infty. \quad (\text{III.17})$$

1.4 The context

1.4.1 Historical comments

Let $N \in \mathbb{N}^*$. Consider the following scalar equation of evolution:

$$\partial_t f_\varepsilon + \frac{1}{\varepsilon} h(v) \cdot \nabla_x f_\varepsilon + \frac{1}{\varepsilon^2} \mathcal{Q} f_\varepsilon = S(t, x, v), \quad f_\varepsilon(t, x, v) \in \mathbb{R} \quad (\text{III.18})$$

where $h : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is some smooth function, \mathcal{Q} is some linear operator acting on L^2 , and $S(t, x, v)$ is some function depending on the variables $(t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}^N$. The unknown is the function $f_\varepsilon(t, x, v)$. Depending on the choice of \mathcal{Q} , the equation (III.18) can be the neutron equation [2], the Fokker-Planck equation [16] or the Boltzmann transport equation [33]. In this context, it is well-known that the family $\{f_\varepsilon\}_{\varepsilon \in]0, 1]}$ has a weak limit, say f_0 as ε goes to 0. In general, the expression f_0 satisfies an equation implying a *drift-diffusion* term of the form $-\operatorname{div}_x(D \nabla_x \cdot)$ where D is some squared matrix depending on the data. The proofs of the related statements rely strongly on the structure of the collision operator \mathcal{Q} which is either a bounded operator or a self-adjoint operator on some weighted version of L^2 . When the operator is less regular or when there is a lack of symmetries [15], the convergence concerns only the mean value ϱ_ε with respect to v , called the density. For some function ϱ_0 satisfying adequate restrictions, we have:

$$\varrho_\varepsilon(t, x) := \int_{\mathbb{R}^N} f_\varepsilon(t, x, v) dv \rightarrow \varrho_0(t, x). \quad (\text{III.19})$$

When looking at the structure of $\mathcal{L}^a := {}^t(\mathcal{L}_1^a, \mathcal{L}_2^a)$, there is some analogy with (III.18). Indeed, the expression $\mathcal{L}^a(\varepsilon, v_\varepsilon)$ can be decomposed into:

$$\mathcal{L}^a(\varepsilon, v_\varepsilon) := \partial_t v_\varepsilon + \frac{1}{\varepsilon} \mathcal{T} v_\varepsilon + \frac{1}{\varepsilon^2} \mathcal{Q} v_\varepsilon + \mathcal{L} \mathcal{L}(\varepsilon) v_\varepsilon + \varepsilon^{M-2} \mathcal{N} \mathcal{L}(\varepsilon, v_\varepsilon) \quad (\text{III.20})$$

with:

$$\begin{aligned} \mathcal{Q} &= \begin{pmatrix} -\mu \partial_{\theta\theta} & 0 \\ \partial_\theta h & -\mu \partial_{\theta\theta} \end{pmatrix}, & \mathcal{T} &:= \begin{pmatrix} h \partial_y - \lambda \partial_{\theta\theta} & 0 \\ 0 & h \partial_y \end{pmatrix}, \\ \mathcal{L} \mathcal{L}(\varepsilon) &:= \begin{pmatrix} -\mu \partial_{yy} & -\lambda \partial_{\theta y} \\ -\lambda \partial_{\theta y} & -(\mu + \varepsilon \lambda) \partial_{yy} \end{pmatrix}, & \mathcal{N} \mathcal{L}(\varepsilon, v_\varepsilon) &:= \begin{pmatrix} v_\varepsilon^1 \partial_\theta v_\varepsilon^1 + \varepsilon v_\varepsilon^2 \partial_y v_\varepsilon^1 \\ v_\varepsilon^1 \partial_\theta v_\varepsilon^2 + \varepsilon v_\varepsilon^2 \partial_y v_\varepsilon^2 \end{pmatrix}. \end{aligned} \quad (\text{III.21})$$

There are many analogies between (III.18) and (III.20). In both cases, the hierarchy with respect to the negative powers of ε (namely ε^{-2} , ε^{-1} and ε^0) is the same, with in factor

operators sharing analogous structures. Also, the mean value operation $\langle v_\varepsilon \rangle$ is when considering (III.20) what replaces the integration with respect to v at the level of (III.19). However, there are two important differences when comparing (III.18) and (III.20):

- The equation (III.20) is a system ($v_\varepsilon \in \mathbb{R}^2$). When dealing only with the singular part $\varepsilon^{-1}\mathcal{T} + \varepsilon^{-2}\mathcal{Q}$, this problem can be circumvented by first solving the equation on v_ε^1 and then by plugging the result into the equation on v_ε^2 . However, once the influences of the contributions \mathcal{LL} or \mathcal{NL} are incorporated, such strong decoupling is no more available. When dealing with the full system (III.20), the discussion must necessarily take into account *vectorial aspects*.

- The operator \mathcal{Q} of (III.20) is neither selfadjoint nor bounded (on L^2). Up to some extent, it can be viewed as a non selfadjoint perturbation of the selfadjoint action $-\mu\partial_{\theta\theta} I$. Still, we can compute the point spectrum $\sigma_P(\mathcal{Q})$ of $\mathcal{Q} : L^2(\mathbb{T}; \mathbb{R}^2) \longrightarrow H^{-2}(\mathbb{T}; \mathbb{R}^2)$. We find that:

$$\sigma_P(\mathcal{Q}) := \{ \delta \in \mathbb{C}; \mathcal{Q} - \delta I \text{ is not injective} \} = \{ \mu n^2; n \in \mathbb{N} \}.$$

From the point of view of central variety theorems, the presence of a (point) spectral gap between the eigenvalue 0 and the other (positive) eigenvalues indicates that there is a separation between two types of behaviours in time, a *slow* one and a *fast* decaying one, for instance in the spirit of [24, 33]. Of course, such a separation is due to the presence of $-\mu\partial_{\theta\theta} I$ inside \mathcal{Q} . Again, the influence of this dissipation term is what relates (III.18) and (III.20).

In other respects, singular systems like (III.20) have been studied in a purely hyperbolic context, that is when $\mu = \lambda_\varepsilon = 0$. Then, the discussion is based on tools coming from supercritical nonlinear geometric optics [1, 9, 11].

The asymptotic analysis of (III.20) under the assumptions retained here is clearly at the interface of what is done in [2, 15, 33] and [1, 9, 11]. Nevertheless, it needs to develop a specific approach which is the matter of the current contribution. In the next paragraph, we give a few indications of our strategy.

1.4.2 Heuristical description on the expansion

Our analysis of \mathcal{L}^a is based on a discrete Fourier decomposition with respect to $\theta \in \mathbb{T}$. We can expand h as well as $v = {}^t(v^1, v^2)$ into Fourier Series:

$$h = \sum_{k \in \mathbb{Z}^*} h_k e^{ik\theta}, \quad v^j = \sum_{k \in \mathbb{Z}} v_k^j e^{ik\theta}, \quad j \in \{1, 2\}.$$

Introduce the following linear map:

$$\begin{aligned} \tilde{\Pi} : L^2(\mathbb{T}; \mathbb{R}^2) &\longrightarrow L^2(\mathbb{T}; \mathbb{R}^2) \\ v = {}^t(v^1, v^2) &\longmapsto {}^t\left(v_0^1, v_0^2 - i \sum_{k \in \mathbb{Z}^*} h_k \mu^{-1} k^{-1} v_0^1 e^{ik\theta}\right). \end{aligned}$$

The application $\tilde{\Pi}$ is clearly a projector onto the kernel of \mathcal{Q} . Retain that:

$$\tilde{\Pi} \circ \tilde{\Pi} = \tilde{\Pi}, \quad \tilde{\Pi} L^2 = \ker \mathcal{Q}, \quad \dim(\ker \mathcal{Q}) = 2.$$

- **a** • To understand the action of the several operators in \mathcal{L}^a defined in (III.20), a first approach is to consider the simplified equation:

$$\partial_t \tilde{v}_\varepsilon + \varepsilon^{-2} \mathcal{Q} \tilde{v}_\varepsilon = 0, \quad \tilde{v}_\varepsilon(0, \theta) = \sum_{k \in \mathbb{Z}} \tilde{v}_k(0) e^{ik\theta}. \quad (\text{III.22})$$

The corresponding solution $\tilde{v}_\varepsilon = {}^t(\tilde{v}_\varepsilon^1, \tilde{v}_\varepsilon^2)$ involves components \tilde{v}_ε^j which can be put in the form

$$\tilde{v}_\varepsilon^j(t) = \sum_{k \in \mathbb{Z}} \tilde{v}_k^j\left(t, \frac{t}{\varepsilon^2}\right) e^{ik\theta}, \quad \tau := \frac{t}{\varepsilon^2}, \quad j \in \{1, 2\}.$$

For $k = 0$, we find that $\tilde{v}_0^1(t, \tau) = \tilde{v}_0^1(0)$ and:

$$\tilde{v}_0^2(t, \tau) = \tilde{v}_0^2(0) - i \sum_{p \in \mathbb{Z}^*} h_p \mu^{-1} p^{-1} \tilde{v}_{-p}^1(0) + i \sum_{p \in \mathbb{Z}^*} h_p \mu^{-1} p^{-1} \tilde{v}_{-p}^1(0) e^{-\mu p^2 \tau}.$$

The second (constant) term in $\tilde{v}_0^2(t, \tau)$ is in general non zero and it comes from contributions inside $\tilde{v}_\varepsilon(0, \cdot)$ which are polarized according to $(I - \tilde{\Pi}) L^2$. Thus, even if $\tilde{v}_\varepsilon(0, \cdot)$ presses only on the positive point spectrum, the corresponding solution \tilde{v}_ε is not necessarily exponentially decreasing in time. We can see here a first effect of the nonselfadjoint part inside \mathcal{Q} .

For $k \in \mathbb{Z}^*$, noting $\aleph := \{p \in \mathbb{Z}^*; p \neq k, p \neq 2k\}$, we have $\tilde{v}_k^1(t, \tau) = \tilde{v}_k^1(0) e^{-\mu k^2 \tau}$ and:

$$\begin{aligned} \tilde{v}_k^2(t, \tau) = & -i h_k \mu^{-1} k^{-1} \tilde{v}_0^1(0) - 2i k h_{2k} \tilde{v}_{-k}^1(0) \tau e^{-\mu k^2 \tau} \\ & + i \sum_{p \in \aleph} h_p \mu^{-1} (p - 2k)^{-1} \tilde{v}_{k-p}^1(0) e^{-\mu(k-p)^2 \tau} - i \sum_{p \in \aleph} h_p \mu^{-1} (p - 2k)^{-1} \tilde{v}_{k-p}^1(0) e^{-\mu k^2 \tau} \\ & + [\tilde{v}_k^2(0) + i h_k \mu^{-1} k^{-1} \tilde{v}_0^1(0)] e^{-\mu k^2 \tau}. \end{aligned}$$

By bringing together all constant terms (in τ) inside an expression $v_k^s(t, \theta)$ which here does not depend on t , these formulas fit with a decomposition like (III.4).

- **b** • Next, consider the more elaborated model:

$$\partial_t \check{v}_\varepsilon + \varepsilon^{-1} \mathcal{T} \check{v}_\varepsilon + \varepsilon^{-2} \mathcal{Q} \check{v}_\varepsilon = 0, \quad \check{v}_\varepsilon(0, y, \theta) = \sum_{k \in \mathbb{Z}} \check{v}_k(0, y) e^{ik\theta}. \quad (\text{III.23})$$

One can expect that the intermediate singular term $\varepsilon^{-1} \mathcal{T}$ (introduced page 65) produces the scaling t/ε . However, such an effect does not appear here. On the one hand, the contributions polarized according to $(I - \tilde{\Pi}) L^2$ are mainly handled as in paragraph **a**. On the other hand, the $\tilde{\Pi} L^2$ parts disappear by a combination of two arguments:

- Due to the relation $\int_{\mathbb{T}} h h' d\theta = 0$, we can use the following algebraic identity:

$$\tilde{\Pi} \circ \mathcal{T} \circ \tilde{\Pi} \equiv 0. \quad (\text{III.24})$$

- We can absorb the extra term $(I - \tilde{\Pi}) \mathcal{T} \tilde{\Pi}$ through some ellipticity inside \mathcal{Q} . Indeed, in what follows, we seek \check{v}_ε as an expansion of the form $\check{v}_\varepsilon = \check{v}_0 + \varepsilon \check{v}_1 + O(\varepsilon^2)$. Assuming that $\check{v}_0 = \tilde{\Pi} \check{v}_0 \in \ker \mathcal{Q}$, we can observe that:

$$(I - \tilde{\Pi}) (\varepsilon^{-1} \mathcal{T} + \varepsilon^{-2} \mathcal{Q}) (\check{v}_0 + \varepsilon \check{v}_1) = \varepsilon^{-1} [(I - \tilde{\Pi}) \mathcal{T} \tilde{\Pi} \check{v}_0 + (I - \tilde{\Pi}) \mathcal{Q} (I - \tilde{\Pi}) \check{v}_1] + O(1).$$

Now, the idea is to adjust \check{v}_1 conveniently in order to remove the $O(\varepsilon^{-1})$ contribution.

In practice, the implementation of these arguments must be done with care because the different terms which come into play are more tangled than what is indicated above.

Note that a normal form approach (in the spirit of [11]: meaning to change \tilde{v} into $(I + \varepsilon M)\tilde{v}$ for some well adjusted operator M), can be tried to get rid of \mathcal{T} . However, such a method seems not to succeed. There are always remaining $O(\varepsilon^{-1})$ terms and, all things considered, to deal with the actual diagonal form of \mathcal{T} appears to be more suitable.

• **c** • Finally, consider the full system (III.20). Our aim is to describe the asymptotic behaviour of the family $\{v_\varepsilon\}_\varepsilon$ on a time scale of the order $t \simeq 1$. To this end, we have to understand the $O(1)$ contributions brought by the singularity $\varepsilon^{-1}\mathcal{T} + \varepsilon^{-2}\mathcal{Q}$. This singular term is a perturbation of the self adjoint operator $\mathcal{Q}_0 := \varepsilon^{-2}\mu\partial_{\theta\theta}I$. This perturbation is of two types.

- The interactions (at order one) between \mathcal{Q}_0 and $\varepsilon^{-1}\mathcal{T}$ turns out to be the source of some creation of diffusion. The mechanism is similar to the one met in the *drift-diffusion* phenomena. Moreover, $\mathcal{Q}_0 + \varepsilon^{-1}\mathcal{T}$ is a diagonal operator. The components of the velocity are decoupled and the discussion deals more with scalar arguments than with vectorial arguments.
- We perturb $\mathcal{Q}_0 + \varepsilon^{-1}\mathcal{T}$ by $\varepsilon^{-2}\begin{pmatrix} 0 & 0 \\ \partial_\theta h & 0 \end{pmatrix}$ at order 0. The main operator is now \mathcal{Q} which is not selfadjoint. It also induces some strong coupling at order 0 between the two components of the velocity. One aspect of the construction is to prove that this strong coupling do not disrupt the production of dissipation. The discussion has to take into account vectorial aspects and one issue is to match the initial data between the slow profile v_ε^s and the fast profile v_ε^f .

Moreover, we have to determine the effects of \mathcal{LL} and \mathcal{NL} which are of two types. First, the presence of \mathcal{LL} and \mathcal{NL} reinforces the coupling. Secondly, it induces nonlinear interactions which are delicate to deal with. In particular, in the critical case $M = 2$, it becomes necessary to exhibit *transparency phenomena* in order to achieve the analysis.

In this article, we propose (Proposition III.3) and we justify (Proposition III.4) a complete expansion for the family $\{v_\varepsilon\}_\varepsilon$. It is the occasion to analyze precisely the linear features and the non linear aspects alluded above.

1.4.3 Heuristical arguments for the energy estimates

In a second part, we justify that the approximated solution $(q_\varepsilon^a, v_\varepsilon^a)$ converges to the true solution $(q_\varepsilon^e, v_\varepsilon^e)$ and estimates the speed of convergence (see Theorem III.4). Since

$$(q_\varepsilon^e, v_\varepsilon^e) = (q_\varepsilon^a, v_\varepsilon^a) + \varepsilon^r(q_\varepsilon^r, v_\varepsilon^r),$$

we prove accurate estimates of the growth in time over the velocity v_ε^R (Proposition III.21) and the pressure q_ε^R (Proposition III.30). It has two goal:

- First to evaluate the speed of convergence (see Inequalities (III.15)-(III.17)).

- The second and non trivial aspect is to prove estimates on a strip independant of ε (Inequality (III.14)). Arguing by contradiction allows us to pass from the estimates of the growth in time (over $(q_\varepsilon^r, v_\varepsilon^r)$) to the existence of an independant time of existence (see Lemma III.37).

The key estimates are performed on the equations satisfied by $(q_\varepsilon^R, v_\varepsilon^R)$ (see System (III.111)). Those variables can be interpreted as the solution of the linearized operator \mathcal{L} at $(q_\varepsilon^a, v_\varepsilon^a)$ perturbed by some non-linear terms. To underline the difficulties to obtain estimates on a strip independant of ε and emphasize the method we propose to solve the problem, we consider, in this introduction, the same question for a simpler problem. There exists a connection between the variable (p, u) and (q, v) (see page 62). So we can wonder how to obtain estimates for the (p, u) variables. Here $(p_\varepsilon^r, u_\varepsilon^r)$ can be interpreted as the solution of the linearized system (III.3) at ${}^t(0, 0, h(\theta)) + (\varepsilon^\nu q_\varepsilon^a, \varepsilon^M v_\varepsilon^a)$ perturbed by some non-linear terms. We consider, here, the solution $(p_\varepsilon^l, u_\varepsilon^l)$ of the linearized system (III.3) at point ${}^t(0, 0, h(\theta))$:

$$\begin{cases} \partial_t p_\varepsilon^l + \varepsilon^{-1} h \partial_y p_\varepsilon^l & = S_0, \\ \partial_t u_\varepsilon^{1l} + \varepsilon^{-1} h \partial_y u_\varepsilon^{1l} - \tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon}^1 u_\varepsilon^l & = S_1, \\ \partial_t u_\varepsilon^{2l} + \varepsilon^{-1} h \partial_y u_\varepsilon^{2l} + \varepsilon^{-2} \partial_\theta h u_\varepsilon^{1l} - \tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon}^2 u_\varepsilon^l & = S_2, \end{cases} \quad (\text{III.25})$$

for some sources $S := {}^t(S_0, S_1, S_2)$ in $H^\infty(\mathbb{T} \times \mathbb{R}; \mathbb{R}^3)$. It is a parabolic-hyperbolic system singular in ε .

•**Purely hyperbolic approach**• We first consider that $\lambda_\varepsilon = \mu = 0$ so that the dissipation vanishes. We perform classical L^2 -estimates on Equation (III.25):

$$\begin{aligned} \frac{1}{2} \partial_t \left\| (p_\varepsilon^l, u_\varepsilon^l) \right\|_{L^2}^2 &\leq \frac{3}{2} \left\| (p_\varepsilon^l, u_\varepsilon^l) \right\|_{L^2}^2 + \frac{1}{2} \|S\|_{L^2}^2 + \varepsilon^{-2} \left| \int_{\mathbb{T} \times \mathbb{R}} \partial_\theta h u_\varepsilon^{1l} u_\varepsilon^{2l} d\theta dy \right| \\ &\leq \frac{1}{2} (3 + \varepsilon^{-2} \|\partial_\theta h\|_{L^\infty}) \left\| (p_\varepsilon^l, u_\varepsilon^l) \right\|_{L^2}^2 + \frac{1}{2} \|S\|_{L^2}^2. \end{aligned}$$

So to say, applying the Gronwall lemma we obtain a L^2 control over the solution:

$$\left\| (p_\varepsilon^l, u_\varepsilon^l) \right\|_{L^2} (t, \cdot) \lesssim e^{C_\varepsilon t} \sup_{t \in [0, T_\varepsilon^l]} \|S\|_{L^2}, \quad \text{with } C_\varepsilon \leq C (1 + \varepsilon^{-2} \|\partial_\theta h\|_{L^\infty}).$$

Yet, it indicates that we can only control the solution for time of order ε^{-2} . In particular for bounded time the solution can exponentially increase with t .

An other issue is the control over $(q_\varepsilon^R, v_\varepsilon^R)$ in the H^1 -norm. Indeed, if we consider the singular transport equation:

$$\begin{cases} \partial_t v + \varepsilon h(\theta) \partial_y v = 0, \\ v|_{t=0} = v_0, \end{cases}$$

the solution is explicit $v = v_0(\theta, y - \varepsilon^{-1} t h(\theta))$. In particular each time we derivate with respect to θ , we loss a power of ε . To compensate it, we have to introduce anisotropic Sobolev spaces (defined page 62) for both the velocity and the pressure.

•**Parabolic-hyperbolic approach**• To go further in time, we have to consider the dissipation. Of course, $\mathcal{P}_{\varepsilon, \lambda_\varepsilon}$ is a positive operator. Here one can prove some coercive estimates.

There exists a positive constant c such that for any function $f \in H^1(\mathbb{T} \times \mathbb{R})$,

$$\forall \varepsilon \in]0, 1], \quad -\left\langle \tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon} f, f \right\rangle \geq c \left(\|\varepsilon^{-1} \partial_\theta f\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 + \|\partial_y f\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 \right) := \Phi_\varepsilon(\nabla, f). \quad (\text{III.26})$$

It has two consequences:

- At fixed ε , we should obtain a regularization of the solution. The velocity u_ε^l is in $L_t^2 H_{\theta, y}^1$ (see Inequality (III.123)). This is the *regularization phenomena*.
- Considering the dependency in ε , this new estimate brings a better control. The family $\{u_\varepsilon^l\}_\varepsilon$ is bounded $L_t^2 H_{\theta, y}^1$. The dissipation should also absorb some singular term in ε :

$$\varepsilon^{-1} \left| \varepsilon^{-1} \int_{\mathbb{T} \times \mathbb{R}} \partial_\theta h v_\varepsilon^{1l} v_\varepsilon^{2l} d\theta dy \right| \lesssim \varepsilon^{-1} \left(\|h\|_{L^\infty}^2 \|v_\varepsilon^l\|_{L^2}^2 + \|\varepsilon^{-1} \partial_\theta v_\varepsilon^l\|_{L^2}^2 \right).$$

Thus the worst term seems to be singular of order one (in ε) instead of being singular of order two (in ε).

What can be underlined here is that the pressure is still estimated in anisotropic Sobolev spaces whereas we hope to obtain estimates over the velocity in the classical Sobolev spaces (thanks to the dissipation which control singular terms). It indicates that the estimate over the velocity and the pressure have to be done separately.

Thus in Subsection 3.1 we first estimate the velocity. Then we plug the estimates obtain for the velocity into the equation of the pressure to deal with the case of the pressure in Subsection 3.2. Of course when dealing with the complete System (III.111) this computation does not seem obvious. Some arguments must be added. We detail again the method in the general case at the beginning of Section 3.

Here, the addition of the parabolic aspect in the discussion still does not allow us to obtain a control over $(p_\varepsilon^l, u_\varepsilon^l)$ for time of order one. Some technics have to be developed.

• **Singular change of unknowns** • To keep on desingularizing the term $\varepsilon^{-2} \partial_\theta h u_\varepsilon^{1l}$ we consider the change of unknowns:

$$\tilde{q}_\varepsilon^l := q_\varepsilon^l, \quad \tilde{u}_\varepsilon^{1l} := u_\varepsilon^{1l}, \quad \tilde{u}_\varepsilon^{2l} := \varepsilon u_\varepsilon^{2l}.$$

The system (III.25) becomes:

$$\begin{cases} \partial_t \tilde{p}_\varepsilon^l + \varepsilon^{-1} h \partial_y \tilde{p}_\varepsilon^l & = S_0, \\ \partial_t \tilde{u}_\varepsilon^{1l} + \varepsilon^{-1} h \partial_y \tilde{u}_\varepsilon^{1l} - \mathcal{Q}_{\varepsilon, \lambda_\varepsilon}^1 \tilde{u}_\varepsilon^l & = S_1, \\ \partial_t \tilde{u}_\varepsilon^{2l} + \varepsilon^{-1} h \partial_y \tilde{u}_\varepsilon^{2l} + \varepsilon^{-1} \partial_\theta h \tilde{u}_\varepsilon^{1l} - \mathcal{Q}_{\varepsilon, \lambda_\varepsilon}^2 \tilde{u}_\varepsilon^l & = \varepsilon S_2. \end{cases} \quad (\text{III.27})$$

where the operator $\mathcal{Q}_{\varepsilon, \lambda_\varepsilon}$ is defined in Equation (3). It has two consequences:

- The singular term $\varepsilon^{-2} \partial_\theta h u_\varepsilon^{1l}$ is turn into $\varepsilon^{-1} \partial_\theta h \tilde{u}_\varepsilon^{1l}$ (what is desired).

- However, the dissipation is turns into the operator $\mathcal{Q}_{\varepsilon, \lambda_\varepsilon}$. It can no longer satisfies Inequality (III.26). Assuming λ_ε goes to 0 fast enough, it is still true (*c.f.* Lemma III.25). Indeed performing the same estimates for system (III.27) as the one done in the previous case should lead to a control over $(\tilde{p}_\varepsilon^R, \tilde{u}_\varepsilon^R)$ in L^2 -norm for time of order one ($t \approx 1$).

In Section 3, we justify that those heuristical arguments work for the complete System (III.111). Some technical arguments must be added to deal with the complete System (III.111). Indeed, it is obviously nonlinear and coupled. Nonlinear terms has to be studied carefully. The problem with the coupling between the velocity and the pressure appears at the level of the estimate over the term

$$-\frac{C\varepsilon^{2\nu-M-R-2}}{2} {}^t(\partial_\theta, \varepsilon \partial_y) (q_\varepsilon^a + \varepsilon^R q_\varepsilon^R)^2,$$

in Equation (III.111). ν has to be large enough so that the pressure does not interfere too much with the velocity. Of course an other issue is that the pressure is only estimated in the anisotropic Sobolev spaces. We can go back to the classical Sobolev spaces using the equivalence of norms (III.106). It has a cost in power of ε for each derivatives to estimate. It explains why we lose $(m+3)$ precision in the definition of w_m (see Equation (III.13)).

1.4.4 Contents

What follows is divided in two main parts: Section 2 and Section 3.

The Section 2 is devoted to the construction of the approximated solutions $(q_\varepsilon^a, v_\varepsilon^a)$. The first step is to show the Proposition III.1.

- In this purpose, the paragraphs 2.1.1, 2.1.2 and 2.1.3 deal with the velocity field v_ε^a , that is with the equation $\mathcal{L}^a(\varepsilon, v_\varepsilon^a) = O(\varepsilon^N)$. In the part 2.1.1, we first neglect all non linear effects by assuming that M is large enough with respect to N (namely that $M-2 > N+1$). The goal is to make clear, already in a linear context, how the singular terms can be managed.
- In the paragraph 2.1.2, we only assume that $M \geq 2$. The analysis goes along the same lines with however modifications to do when computing the source terms. Also, the limit case $M=2$ is special because in this situation the non linear terms can interfere at leading order.
- In the paragraph 2.1.3, we are able to exhibit the control (III.8).

The pressure component q_ε^a is incorporated at the level of subsection 2.2. Then, the complete construction of $(q_\varepsilon^a, v_\varepsilon^a)$ can be achieved in the form of Proposition III.3.

The Section 3 is concerned with energy estimates. In the subsection 3.1, we look at the equations \mathcal{L}_1 and \mathcal{L}_2 . To this end, we crucially need the properties brought by the dissipation. In the subsection 3.2, we inject the informations which have been obtained at the level of \mathcal{L}_0 . By this way, we can deduce controls concerning the pressure component. Finally, in the subsection 3.3, we prove all the estimates which are stated in the Proposition III.4.

2 Construction of approximated solutions

We make here the assumption $\lambda_\varepsilon := \varepsilon\lambda$ with $\lambda > 0$. We simply write $\tilde{\mathcal{P}}_\varepsilon$ in place of $\tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon}$. This section is dedicated to the proof of Propositions III.1, III.2 and III.3.

2.1 Approximated solutions for \mathcal{L}^a - Proof of Proposition III.1

In the paragraph 2.1.1, we first consider the case $M - 2 > N + 1$. This simplifying assumption amounts to forget all nonlinear terms, that is to concentrate on the linear aspects of the analysis. Nonlinear influences are incorporated at the level of Subsection 2.1.2 where we only suppose that $M \geq 2$. Finally, in paragraph 2.1.3, we derive various estimates concerning approximated solutions, especially the crucial control (III.8).

2.1.1 The case $M - 2 > N + 1$.

As indicated in the title, we fix two integers M and N such that $M - 2 > N + 1$. It follows that it is not necessary to take into account the $O(\varepsilon^{M-2})$ nonlinear contributions inside $\mathcal{L}^a(\varepsilon, v_\varepsilon)$ in order to recover (III.8). Nonlinear terms do not interfere in this paragraph 2.1.1.

Since the equation \mathcal{L}^a is linear at the $O(\varepsilon^N)$ precision under study, we can adopt the following strategy. We construct independently expansions

$$v_\varepsilon^s(t, y, \theta) = \sum_{k=0}^{N+1} \varepsilon^k v_k^s(t, y, \theta), \quad v_\varepsilon^f\left(\frac{t}{\varepsilon^2}, y, \theta\right) = \sum_{k=0}^{N+1} v_k^f\left(\frac{t}{\varepsilon^2}, y, \theta\right) \quad (\text{III.28})$$

which satisfy the condition (III.8) on any time interval $[0, T]$. The sum $v_\varepsilon^a = v_\varepsilon^s + v_\varepsilon^f$ is of course a solution of $\mathcal{L}^a(\varepsilon, v_\varepsilon^a) = O(\varepsilon^N)$. The difficulty is mainly to show that all initial data prescribed in (III.7) can indeed be obtained by this way.

By convention, we define $v_k^s \equiv 0$ and $v_k^f \equiv 0$ for $k = -3$, $k = -2$, and $k = -1$. We plug v_ε^s and v_ε^f as indicated in (III.28) inside $\mathcal{L}^a(\varepsilon, \cdot)$ to get two different cascades of equations. The one corresponding to the time scale t can be put in the form, for $k \in \llbracket 0, N + 3 \rrbracket$:

$$\partial_t v_{1,k-2}^s - \mu \partial_{yy} v_{1,k-2}^s = -h \partial_y v_{1,k-1}^s + \lambda \partial_{\theta\theta} v_{1,k-1}^s + \lambda \partial_{\theta y} v_{2,k-2}^s + \mu \partial_{\theta\theta} v_{1,k}^s, \quad (\text{III.29a})$$

$$\begin{aligned} \partial_t v_{2,k-2}^s - \mu \partial_{yy} v_{2,k-2}^s &= -h \partial_y v_{2,k-1}^s - \partial_\theta h v_{1,k}^s + \mu \partial_{\theta\theta} v_{2,k}^s \\ &\quad + \lambda \partial_{\theta y} v_{1,k-2}^s + \lambda \partial_{yy} v_{2,k-3}^s. \end{aligned} \quad (\text{III.29b})$$

The other (corresponding to the time scale τ) can be written for $k \in \llbracket 0, N + 1 \rrbracket$:

$$\partial_\tau v_{1,k}^f - \mu \partial_{\theta\theta} v_{1,k}^f = -h \partial_y v_{1,k-1}^f + \lambda \partial_{\theta\theta} v_{1,k-1}^f + \mu \partial_{yy} v_{1,k-2}^f + \lambda \partial_{\theta y} v_{2,k-2}^f, \quad (\text{III.30a})$$

$$\begin{aligned} \partial_\tau v_{2,k}^f - \mu \partial_{\theta\theta} v_{2,k}^f &= -h \partial_y v_{2,k-1}^f - \partial_\theta h v_{1,k}^f + \mu \partial_{yy} v_{2,k-2}^f \\ &\quad + \lambda \partial_{\theta y} v_{1,k-2}^f + \lambda \partial_{yy} v_{2,k-3}^f. \end{aligned} \quad (\text{III.30b})$$

In view of (III.7), we have also to impose:

$$v_k^s(0, \theta, y) + v_k^f(0, \theta, y) = v_k^0(\theta, y), \quad \forall k \in \llbracket 0, N + 1 \rrbracket. \quad (\text{III.31})$$

Proposition III.5 (Solving the systems III.29 and III.30 together with III.31). *Fix any time $T \in \mathbb{R}_+^*$, any number $\delta \in]0, \mu[$ and, for $k \in \{0, \dots, N+1\}$, any functions $v_k^0 \in \mathcal{H}^\infty(\mathbb{R} \times \mathbb{T})$. Then, the set of conditions (III.29*), (III.30*) and (III.31) has a unique solution satisfying:*

$$(v_k^s, v_k^f) \in \mathcal{H}_T^\infty \times \mathcal{E}_\delta^\infty, \quad \forall k \in \{0, \dots, N+1\}. \quad (\text{III.32})$$

Moreover, the component v_k^s can be identified through the homogenized equation (III.9).

The proof of Proposition III.5 is given at the end of this paragraph 2.1.1. It is obtained by induction on the size of N . Thus, mark the hypothesis:

$$\mathcal{H}(N) : \text{ " The Proposition III.5 is verified up to the integer } N \text{ "}. \quad (\text{III.33})$$

To go from N up to $N+1$, we will need a succession of lemmas which are produced below. In what follows, given a symbol $\mathcal{V} \in \{W^{m,p}, H^s, \mathcal{W}_T^{m,s}, \mathcal{H}_T^s, \mathcal{E}_\delta^s\}$, we will manipulate functions $f(\theta, y) \in \mathcal{V}(\mathbb{T} \times \mathbb{R})$. We will often decompose f as indicated in paragraph 1.2.3 into its mean and oscillating parts according to

$$f^\parallel(y) := \langle f(\cdot, y) \rangle \in \mathcal{V}^\parallel := \Pi \mathcal{V}(\mathbb{T} \times \mathbb{R}), \quad f^\perp(\theta, y) := f^*(\theta, y) \in V^\perp := (I - \Pi) \mathcal{V}(\mathbb{T} \times \mathbb{R}). \quad (\text{III.34})$$

To signal that we consider functions $f(\theta, y)$ which do not depend on $\theta \in \mathbb{T}$ ($\Pi f = f$) or whose mean value is zero ($\Pi f = 0$), we will use respectively (as above) the marks \parallel and \perp . By extension, when dealing with some operator P , we will note

$$P^\parallel := P\Pi, \quad P^\perp := P(I - \Pi). \quad (\text{III.35})$$

Be careful, in the case of operators, the composition by Π and $I - \Pi$ is put *on the right*.

• **A consequence of the (point) spectral gap** • The system (III.30) is made of two evolution equations of parabolic type, based on $\partial_\tau - \mu \partial_{\theta\theta}$. This falls under the following framework.

Lemma III.6 (Fast decreasing under a polarization condition). *Let $m \in \mathbb{N}$ and $\delta \in]0, \mu[$. Select $w_0 \in H^{m+2}(\mathbb{T} \times \mathbb{R})$ and $S_0 \in (\mathcal{E}_\delta^{m+2})^\perp(\mathbb{T} \times \mathbb{R})$, that is such that $\Pi S_0 = 0$. Consider the initial value problem:*

$$\begin{cases} \partial_\tau w - \mu \partial_{\theta\theta} w = S_0, & w|_{\tau=0} = w_0. \end{cases} \quad (\text{III.36})$$

For all $T \in \mathbb{R}_+$, there is a unique solution $w \in C^1([0, T]; H^m(\mathbb{T} \times \mathbb{R}))$ to the Cauchy problem (III.36). Moreover, if the initial data is well prepared in the sense that $\Pi w_0 = 0$, then $\Pi w = 0$ for all $t \in [0, T]$ and $w \in \mathcal{E}_\delta^m(\mathbb{T} \times \mathbb{R})$.

The proof of this Lemma III.6 is very easy. It will not be detailed.

• **Interpretation of the system (III.29)** • The Lemma below is intended to look at the system (III.29) otherwise. Indeed, there is a difficulty when dealing with (III.29) since the knowledge of v_{k-2}^s seems to require the determination of v_{k-1}^s and v_k^s , that is the identification of terms v_j^s with indices j greater than $k-2$. An important remark is that such a dependence disappear when the projector Π is applied. This fact can be formulated in the following way.

Lemma III.7 (Linear homogenization). *Assume that the expressions v_k^s with $k \in \llbracket 0, N+3 \rrbracket$ are solutions of the system (III.29). Then, for all $k \in \{0, \dots, N+1\}$, the part $\Pi v_k^s = {}^t(\Pi v_{1,k}^s, \Pi v_{2,k}^s)$ is a solution of*

$$\begin{cases} \partial_t \Pi v_{1,k}^s - \left(\mu + \frac{1}{\mu} \Pi((\partial_\theta^{-1} h)^2) \right) \partial_{yy} \Pi v_{1,k}^s = S_{1,k}^{l//} + P_1^{l\perp} (I - \Pi) v_{1,k}^s, \\ \partial_t \Pi v_{2,k}^s - \left(\mu + \frac{1}{\mu} \Pi((\partial_\theta^{-1} h)^2) \right) \partial_{yy} \Pi v_{2,k}^s = S_{2,k}^{l//} + P_2^{l\perp} (I - \Pi) v_{1,k}^s \\ \quad + P_2^{l//} \Pi v_{1,k}^s + Q_2^{l\perp} (I - \Pi) v_{2,k}^s, \end{cases} \quad (\text{III.37})$$

where the definitions of the operators $P_1^{l\perp}$, $P_2^{l\perp}$, $P_2^{l//}$ and $Q_2^{l\perp}$ involve only h , whereas the source terms $S_{1,k}^{l//}$ and $S_{2,k}^{l//}$ depend only on the v_j^s with $j \leq k-1$, see (III.38)–(III.45).

Proof of the Lemma III.7. When applying Π on (III.29), there are many cancellations. It remains

$$\begin{cases} \partial_t \Pi v_{1,k}^s - \mu \partial_{yy} \Pi v_{1,k}^s = -\Pi(h \partial_y v_{1,k+1}^s), \\ \partial_t \Pi v_{2,k}^s - \mu \partial_{yy} \Pi v_{2,k}^s = -\Pi(h \partial_y v_{2,k+1}^s) - \Pi(\partial_\theta h v_{1,k+2}^s) + \lambda \partial_{yy} \Pi v_{2,k-1}^s. \end{cases}$$

The matter now is to transform $\Pi(h \partial_y v_{1,k+1}^s)$, $\Pi(h \partial_y v_{2,k+1}^s)$ and $\Pi(\partial_\theta h v_{1,k+2}^s)$. To this end, we will use many integrations by parts and permutations of the derivatives ∂_θ and ∂_y .

◦ Study of $\Pi(h \partial_y v_{1,k+1}^s)$. Note first that:

$$\Pi(h \partial_y v_{1,k+1}^s) = \Pi(\partial_{\theta\theta}^{-2}(h) \partial_y \partial_{\theta\theta} v_{1,k+1}^s).$$

Using (III.29a) to extract $\partial_{\theta\theta} v_{1,k+1}^s$. This yields:

$$\begin{aligned} \Pi(h \partial_y v_{1,k+1}^s) &= \frac{1}{\mu} \Pi(\partial_{\theta\theta}^{-2}(h) \partial_y (\partial_t v_{1,k-1}^s - \mu \partial_{yy} v_{1,k-1}^s - \lambda \partial_{\theta y} v_{2,k-1}^s)) \\ &\quad + \frac{1}{\mu} \Pi(h \partial_{\theta\theta}^{-2}(h) \partial_{yy} v_{1,k}^s) - \frac{\lambda}{\mu} \Pi(\partial_{\theta\theta}^{-2}(h) \partial_{\theta\theta} \partial_y (I - \Pi) v_{1,k}^s). \end{aligned}$$

Then,

$$\begin{aligned} \Pi(h \partial_{\theta\theta}^{-2}(h) \partial_{yy} v_{1,k}^s) &= \Pi(h \partial_{\theta\theta}^{-2}(h) \partial_{yy} \Pi v_{1,k}^s) + \Pi(h \partial_{\theta\theta}^{-2}(h) \partial_{yy} (I - \Pi) v_{1,k}^s) \\ &= -\Pi((\partial_\theta^{-1} h)^2) \partial_{yy} \Pi v_{1,k}^s + \Pi(h \partial_{\theta\theta}^{-2}(h) \partial_{yy} (I - \Pi) v_{1,k}^s). \end{aligned}$$

Finally,

$$\Pi(h \partial_y v_{1,k+1}^s) = -\frac{1}{\mu} \Pi((\partial_\theta^{-1} h)^2) \partial_{yy} \Pi v_{1,k}^s - S_{1,k}^{l//} - P_1^{l\perp} (I - \Pi) v_{1,k}^s$$

with:

$$S_{1,k}^{l//} := -\frac{1}{\mu} \partial_{\theta\theta}^{-2}(h) \partial_y (\partial_t v_{1,k-1}^s - \mu \partial_{yy} v_{1,k-1}^s - \lambda \partial_{\theta y} v_{2,k-1}^s), \quad (\text{III.38})$$

$$P_1^{l//} f := \frac{\lambda}{\mu} \Pi(\partial_{\theta\theta}^{-2}(h) \partial_{\theta\theta y} f) - \frac{1}{\mu} \Pi(h \partial_{\theta\theta}^{-2}(h) \partial_{yy} f). \quad (\text{III.39})$$

◦ Let us compute $\Pi(h \partial_y v_{2,k+1}^s) = \Pi(\partial_{\theta\theta}^{-2}(h) \partial_y \partial_{\theta\theta} v_{2,k+1}^s)$. Using the Equation (III.29b) and after two integration by parts with respect to θ in the second line below, we can deduce:

$$\begin{aligned} \Pi(h \partial_y v_{2,k+1}^s) &= \frac{1}{\mu} \Pi(\partial_{\theta\theta}^{-2}(h) \partial_y (\partial_t v_{2,k-1}^s - \mu \partial_{yy} v_{2,k-1}^s - \lambda \partial_{\theta y} v_{1,k-1}^s - \lambda \partial_{yy} v_{2,k-2}^s)) \\ &\quad + \frac{1}{\mu} \Pi(h \partial_{\theta\theta}^{-2}(h) \partial_{yy} v_{2,k}^s) + \frac{1}{\mu} \Pi(\partial_{\theta} h \partial_{\theta\theta}^{-2}(h) \partial_y v_{1,k+1}^s). \end{aligned}$$

Furthermore,

$$\Pi(h \partial_{\theta\theta}^{-2}(h) \partial_{yy} v_{2,k}^s) = -\Pi((\partial_{\theta}^{-1} h)^2) \partial_{yy} \Pi v_{2,k}^s + \Pi(h \partial_{\theta\theta}^{-2}(h) \partial_{yy} (I - \Pi) v_{2,k}^s),$$

and

$$\begin{aligned} \Pi(\partial_{\theta} h \partial_{\theta\theta}^{-2}(h) \partial_y v_{1,k+1}^s) &= \Pi(\partial_{\theta} h \partial_{\theta\theta}^{-2}(h) \partial_y \Pi v_{1,k+1}^s) + \Pi(\partial_{\theta} h \partial_{\theta\theta}^{-2}(h) \partial_y (I - \Pi) v_{1,k+1}^s) \\ &= 0 + \Pi(\partial_{\theta} h \partial_{\theta\theta}^{-2}(h) \partial_y (I - \Pi) v_{1,k+1}^s). \end{aligned}$$

Using Equation (III.29a), we obtain:

$$\begin{aligned} \Pi(\partial_{\theta} h \partial_{\theta\theta}^{-2}(h) \partial_y v_{1,k+1}^s) &= \frac{1}{\mu} \Pi(\partial_{\theta} h \partial_{\theta\theta}^{-2}(h) \partial_y \partial_{\theta\theta}^{-2}(I - \Pi) (\partial_t v_{1,k-1}^s - \mu \partial_{yy} v_{1,k-1}^s - \lambda \partial_{\theta y} v_{2,k-1}^s)) \\ &\quad + \frac{1}{\mu} \Pi(\partial_{\theta} h \partial_{\theta\theta}^{-2}(h) \partial_y \partial_{\theta\theta}^{-2}(I - \Pi) (h \partial_y v_{1,k}^s - \lambda \partial_{\theta\theta} (I - \Pi) v_{1,k}^s)). \end{aligned}$$

We finally write

$$\begin{aligned} \Pi(h \partial_y v_{2,k+1}^s) &= -\frac{1}{\mu} \Pi((\partial_{\theta}^{-1} h)^2) \partial_{yy} \Pi v_{2,k}^s - S_{2,k}^{l1//} \\ &\quad - P_2^{l1\perp} (I - \Pi) v_{1,k}^s - P_2^{l1//} \Pi v_{1,k}^s - Q_2^{l1\perp} (I - \Pi) v_{2,k}^s, \end{aligned}$$

with

$$\begin{aligned} S_{2,k}^{l1} &:= -\frac{1}{\mu} \partial_{\theta\theta}^{-2}(h) \partial_y (\partial_t v_{2,k-1}^s - \mu \partial_{yy} v_{2,k-1}^s - \lambda \partial_{\theta y} v_{1,k-1}^s - \lambda \partial_{yy} v_{2,k-2}^s) \\ &\quad - \frac{1}{\mu^2} \partial_{\theta} h \partial_{\theta\theta}^{-2}(h) \partial_y \partial_{\theta\theta}^{-2}(I - \Pi) (\partial_t v_{1,k-1}^s - \mu \partial_{yy} v_{1,k-1}^s - \lambda \partial_{\theta y} v_{2,k-1}^s), \end{aligned} \quad (\text{III.40})$$

$$P_2^{l1} f := -\Pi \left(\frac{1}{\mu^2} \partial_{\theta} h \partial_{\theta\theta}^{-2}(h) \partial_y \partial_{\theta\theta}^{-2}(I - \Pi) (h \partial_y f - \lambda \partial_{\theta\theta} f) \right), \quad (\text{III.41})$$

$$Q_2^{l1} f := -\frac{1}{\mu} \Pi(h \partial_{\theta\theta}^{-2}(h) \partial_{yy} f). \quad (\text{III.42})$$

◦ It remains to compute $\Pi(\partial_{\theta} h v_{1,k+2}^s)$. We start with observing that

$$\Pi(\partial_{\theta} h v_{1,k+2}^s) = \Pi(\partial_{\theta}^{-1}(h) \partial_{\theta\theta} v_{1,k+2}^s).$$

Then, using the Equation (III.29a), we can write:

$$\begin{aligned} \Pi(\partial_{\theta} h v_{1,k+2}^s) &= \frac{1}{\mu} \Pi(\partial_{\theta}^{-1}(h) (\partial_t v_{1,k}^s - \mu \partial_{yy} v_{1,k}^s - \lambda \partial_{\theta y} (I - \Pi) v_{2,k}^s)) \\ &\quad - \frac{1}{\mu} \Pi(\partial_{\theta}^{-1}(h) (-h \partial_y v_{1,k+1}^s + \lambda \partial_{\theta\theta} v_{1,k+1}^s)). \end{aligned}$$

As $\Pi(h \partial_\theta^{-1} h) = 0$, we deduce

$$\begin{aligned} \Pi(\partial_\theta h v_{1,k+2}^s) &= \frac{1}{\mu} \Pi(\partial_\theta^{-1}(h)(\partial_t v_{1,k}^s - \mu \partial_{yy} v_{1,k}^s - \lambda \partial_{\theta y}(I - \Pi)v_{2,k}^s)) \\ &\quad - \frac{1}{\mu} \Pi(\partial_\theta^{-1}(h)(-h \partial_y + \lambda \partial_{\theta\theta})(I - \Pi)v_{1,k+1}^s)). \end{aligned}$$

Finally, applying again (III.29a), we obtain

$$\Pi(\partial_\theta h v_{1,k+2}^s) = -S_{2,k}^{l2//} - P_2^{l2\perp} (I - \Pi)v_{1,k}^s - P_2^{l2//} \Pi v_{1,k}^s - Q_2^{l2\perp} (I - \Pi)v_{2,k}^s,$$

with

$$S_{2,k}^{l2} := \frac{1}{\mu^2} \partial_\theta^{-1}(h) (-h \partial_y + \lambda \partial_{\theta\theta}) \partial_{\theta\theta}^{-2} (I - \Pi)(\partial_t v_{1,k-1}^s - \mu \partial_{yy} v_{1,k-1}^s - \lambda \partial_{\theta y} v_{2,k-1}^s) \quad (\text{III.43})$$

$$\begin{aligned} P_2^{l2} f &:= \frac{1}{\mu^2} \Pi(\partial_\theta^{-1}(h) (-h \partial_y + \lambda \partial_{\theta\theta}) \partial_{\theta\theta}^{-2} (I - \Pi)(h \partial_y f - \lambda \partial_{\theta\theta} f)) \\ &\quad - \frac{1}{\mu} \Pi(\partial_\theta^{-1}(h)(\partial_t f - \mu \partial_{yy} f)), \end{aligned} \quad (\text{III.44})$$

$$Q_2^{l2} f := \frac{\lambda}{\mu} \Pi(\partial_\theta^{-1}(h)(\partial_{\theta y} f)). \quad (\text{III.45})$$

Briefly, for all $k \in \mathbb{N}$, we have (III.37) with

$$S_{2,k}^l := S_{2,k}^{l1} + S_{2,k}^{l2} + \lambda \partial_{yy} \Pi v_{2,k-1}^s, \quad P_2^l := P_2^{l1} + P_2^{l2}, \quad Q_2^l := Q_2^{l1} + Q_2^{l2}. \quad \blacksquare$$

• **Analysis of the system (III.37)** • The system (III.37) is clearly an evolution equation of parabolic type. As such, it can be completed by initial datas. But to solve it, we also need to identify the extra terms $(I - \Pi)v_{\star,k}^s$ with $\star \in \{1, 2\}$. To this end, it suffices again to exploit (III.29). Recall that $\mathcal{V} \in \{W^{m,p}, H^s, \mathcal{W}_T^{m,s}, \mathcal{H}_T^s, \mathcal{E}_\delta^s\}$ and define the linear continuous isomorphism

$$\begin{aligned} \Phi : \mathcal{V}(\mathbb{T} \times \mathbb{R}) \times \mathcal{V}(\mathbb{T} \times \mathbb{R}) &\longrightarrow \mathcal{V}^{tot} := \mathcal{V}^\perp(\mathbb{T} \times \mathbb{R}) \times \mathcal{V}^\parallel(\mathbb{R}) \times \mathcal{V}^\perp(\mathbb{T} \times \mathbb{R}) \times \mathcal{V}^\parallel(\mathbb{R}) \\ (f_1, f_2) &\longmapsto ((I - \Pi)f_1, \Pi f_1, (I - \Pi)f_2, \Pi f_2). \end{aligned}$$

By construction, the expression $V_k^s := {}^t \Phi v_k^s$ must be solution of the system

$$\mathcal{A} V_k^s := \begin{pmatrix} \mu \partial_{\theta\theta} & 0 & 0 & 0 \\ -P_1^{l\perp} & P_y & 0 & 0 \\ -T_s & -T_s & \mu \partial_{\theta\theta} & 0 \\ -P_2^{l\perp} & -P_2^{l//} & -Q_2^{l\perp} & P_y \end{pmatrix} V_k^s = \begin{pmatrix} f_{1,k}^\perp \\ f_{1,k}^\parallel \\ f_{2,k}^\perp \\ f_{2,k}^\parallel \end{pmatrix}, \quad V_k^s = \begin{pmatrix} v_{1,k}^{s\perp} \\ v_{1,k}^{s//} \\ v_{2,k}^{s\perp} \\ v_{2,k}^{s//} \end{pmatrix} \quad (\text{III.46})$$

where T_s and P_y are the operators defined by

$$T_s f := (I - \Pi)(\partial_\theta h f), \quad P_y f := \partial_t f - \left(\mu + \frac{1}{\mu} \Pi((\partial_\theta^{-1} h)^2) \right) \partial_{yy} f$$

and where the components $f_{*,k}^*$ can be viewed as source terms (since they depend only on the v_j^s with $j < k$). More precisely, we find $f_{j,k}^{\parallel} := S_{j,k}^{\parallel}$ for $j \in \{1, 2\}$ and

$$f_{1,k}^{l\perp} := (I - \Pi) (\partial_t v_{1,k-2}^s - \mu \partial_{yy} v_{1,k-2}^s - \lambda \partial_{\theta y} v_{2,k-2}^s + h \partial_y v_{1,k-1}^s - \lambda \partial_{\theta\theta} v_{1,k-1}^s), \quad (\text{III.47})$$

$$f_{2,k}^{l\perp} := (I - \Pi) (\partial_t v_{2,k-2}^s - \mu \partial_{yy} v_{2,k-2}^s - \lambda \partial_{\theta y} v_{1,k-2}^s + h \partial_y v_{2,k-1}^s - \lambda \partial_{yy} v_{2,k-3}^s). \quad (\text{III.48})$$

To simplify, we present below the result in a smooth setting.

Lemma III.8 (Solving the system [III.46](#)). *Select a function $V_0^{\parallel} = {}^t(V_0^{1\parallel}, V_0^{2\parallel}) \in H^\infty(\mathbb{R})^2$ and a source term $F = {}^t(F^{1\perp}, F^{1\parallel}, F^{2\perp}, F^{2\parallel}) \in (\mathcal{H}_T^\infty)^{\text{tot}}$. Then, for all $T \in \mathbb{R}_+^*$, the problem*

$$\{ \mathcal{A}V = F, \quad V = {}^t(V^{1\perp}, V^{1\parallel}, V^{2\perp}, V^{2\parallel}), \quad {}^t(V^{1\parallel}, V^{2\parallel})|_{t=0} = {}^t(V_0^{1\parallel}, V_0^{2\parallel}) \} \quad (\text{III.49})$$

has a unique solution V in $(\mathcal{H}_T^\infty)^{\text{tot}}$.

Proof of the Lemma III.8. To solve [III.46](#), the strategy is to argue line after line.

- *First line.* Since the operator $\mu \partial_{\theta\theta} : (\mathcal{H}_T^\infty)^\perp \longrightarrow (\mathcal{H}_T^\infty)^\perp$ is invertible, we can define without ambiguity (and with no choice)

$$V^{1\perp} := \mu^{-1} \partial_{\theta\theta}^{-2} F^{1\perp} \in (\mathcal{H}_T^\infty)^\perp. \quad (\text{III.50})$$

- *Second line.* Observe that $P_1^{l\perp} : \mathcal{H}_T^\infty \longrightarrow (\mathcal{H}_T^\infty)^\perp$. The next component $V^{1\parallel}$ must be a solution of the heat equation (in t and y)

$$\{ P_y V^{1\parallel} = F^{1\parallel} + P_1^{l\perp} V^{1\perp} \in (\mathcal{H}_T^\infty)^{\parallel}, \quad (V^{1\parallel})|_{t=0} = V_0^{1\parallel} \in (\mathcal{H}_T^\infty)^{\parallel} \}. \quad (\text{III.51})$$

Obviously, there is a unique solution on $[0, T]$ of this initial value problem. It does not depend on θ . In other words, it is such that $V(t, \cdot) \in (\mathcal{H}_T^\infty)^{\parallel}$ for all $t \in [0, T]$.

- *Third line.* Since $T_s : \mathcal{H}_T^\infty \longrightarrow (\mathcal{H}_T^\infty)^\perp$ the component $V^{2\perp}$ can be obtained through the formula

$$V^{2\perp} = \mu^{-1} \partial_{\theta\theta}^{-2} (F^{2\perp} + T_s V^{1\perp} + T_s V^{1\parallel}) \in (\mathcal{H}_T^\infty)^\perp. \quad (\text{III.52})$$

- *Fourth line.* We can use the same argument as in the second line. It suffices to check that by definition $P_2^l : \mathcal{H}_T^\infty \longrightarrow (\mathcal{H}_T^\infty)^\perp$ and $Q_2^l : \mathcal{H}_T^\infty \longrightarrow (\mathcal{H}_T^\infty)^\perp$. Then, it remains to solve

$$\{ P_y V^{2\parallel} = F^{2\parallel} + P_2^{l\perp} V^{1\perp} + P_2^{l\parallel} V^{1\parallel} + Q_2^{l\perp} V^{2\perp} \in (\mathcal{H}_T^\infty)^{\parallel}, \quad (V^{2\parallel})|_{t=0} = V_0^{2\parallel} \in (\mathcal{H}_T^\infty)^{\parallel} \}.$$

Note that the triangular structure of \mathcal{A} is crucial in this procedure. ■

• **Analysis of the system [\(III.30\)](#)** • In this paragraph, we consider the linear parabolic system [\(III.30\)](#) which can be associated with some smooth initial data $v_k^f(0, \cdot) \in \mathcal{H}^\infty(\mathbb{T} \times \mathbb{R})$. Classical statements (see for instance [\[26\]](#)) say that the corresponding Cauchy problem has a unique global solution v_k^f such that $v_k^f(t, \cdot) \in \mathcal{H}(\mathbb{T} \times \mathbb{R})$ for all $\tau \in \mathbb{R}_+$. The difficulty is the following. The variable τ is aimed to be replaced by $\varepsilon^{-2} t$ with t fixed and $\varepsilon \rightarrow 0$ and, since the original equation [\(III.3\)](#) contains nonlinearities, we cannot allow any (uncontrolled)

growth with respect to τ . To get round this problem, we will instead require a rapid decay when $\tau \rightarrow +\infty$ but this necessitates $v_k^f(0, \cdot)$ to be selected conveniently.

To see how to adjust $v_k^f(0, \cdot)$, we can interpret (III.30) in the form

$$\mathcal{B} V_k^f := \begin{pmatrix} P_\theta & 0 & 0 & 0 \\ 0 & \partial_\tau & 0 & 0 \\ T_s & T_s & P_\theta & 0 \\ T_f & 0 & 0 & \partial_\tau \end{pmatrix} \begin{pmatrix} g_{1,k}^{l\perp} \\ g_{1,k}^{l\parallel} \\ g_{2,k}^{l\perp} \\ g_{2,k}^{l\parallel} \end{pmatrix}, \quad V_k^f := {}^t\Phi(v_k^f) = \begin{pmatrix} v_{1,k}^{f\perp} \\ v_{1,k}^{f\parallel} \\ v_{2,k}^{f\perp} \\ v_{2,k}^{f\parallel} \end{pmatrix} \quad (\text{III.53})$$

where T_f and P_θ are the operators defined by

$$T_f f := \Pi(\partial_\theta h f), \quad P_\theta f := \partial_\tau f - \mu \partial_{\theta\theta} f$$

and where $g_k^l = {}^t(g_{1,k}^l, g_{2,k}^l)$ can be viewed as a source term (since it depends only on the v_j^f with $j < k$). More precisely, we find that

$$\begin{cases} g_{1,k}^l := (-h \partial_y v_{1,k-1}^f + \lambda \partial_{\theta\theta} v_{1,k-1}^f) + (\mu \partial_{yy} v_{1,k-1}^f + \lambda \partial_{\theta y} v_{2,k-2}^f), \\ g_{2,k}^l := -h \partial_y v_{2,k-1}^f + (\lambda \partial_{\theta y} v_{1,k-2}^f + \mu \partial_{yy} v_{2,k-2}^f) + \lambda \partial_{yy} v_{2,k-3}^f. \end{cases} \quad (\text{III.54})$$

In order to guarantee the fast decaying criterion in τ , we can proceed as described below.

Lemma III.9 (Solving III.53 in the case of a fast decay when τ tends to $+\infty$). *Select functions*

$$V_0^\perp = {}^t(V_0^{1\perp}, V_0^{2\perp}) \in H^\infty(\mathbb{T} \times \mathbb{R})^2, \quad G = {}^t(G^{1\perp}, G^{1\parallel}, G^{2\perp}, G^{2\parallel}) \in (\mathcal{E}_\delta^\infty)^{tot}, \quad \delta \in]0, \mu[.$$

There is a unique expression $V_0^\parallel = {}^t(V_0^{1\parallel}, V_0^{2\parallel}) \in (H^\infty)^\parallel(\mathbb{T} \times \mathbb{R}; \mathbb{R})^2$ which can be determined in function of G through formulas (III.56) and (III.57) such that the Cauchy problem

$$\{\mathcal{B} V = G, \quad V = {}^t(V^{1\perp}, V^{1\parallel}, V^{2\perp}, V^{2\parallel}), \quad V|_{\tau=0} = {}^t(V_0^{1\perp}, V_0^{1\parallel}, V_0^{2\perp}, V_0^{2\parallel})\} \quad (\text{III.55})$$

has a global solution V belonging to the space $(\mathcal{E}_\delta^\infty)^{tot}$.

Proof of the Lemma III.9. The strategy is again to argue line after line.

- *First line.* Just apply the end of Lemma III.6.

- *Second line.* It suffices to take

$$V_0^{1\parallel}(\cdot) := - \int_0^{+\infty} G^{1\parallel}(s, \cdot) ds \in (H^\infty)^\parallel \quad (\text{III.56})$$

in order to recover after integration that $V^{1\parallel} \in (\mathcal{E}_\delta^\infty)^\parallel$ with

$$V^{1\parallel}(\tau, \cdot) = V_0^{1\parallel}(\cdot) + \int_0^\tau G^{1\parallel}(s, \cdot) ds = - \int_\tau^{+\infty} G^{1\parallel}(s, \cdot) ds.$$

- *Third line.* For all $m \in \mathbb{N}$, the operator $T_s : H^m \rightarrow (H^m)^\perp$ is continuous. It follows that T_s sends the functional space $\mathcal{E}_\delta^\infty$ into $(\mathcal{E}_\delta^\infty)^\perp$. Concerning $V^{2\perp}$, the argument is again Lemma III.6 applied this time with the source term $G^{2\perp} - T_s V^{1\perp} - T_s V^{1\parallel} \in (\mathcal{E}_\delta^\infty)^\perp$.

- *Fourth line.* For all $m \in \mathbb{N}$, the operator $T_f : H^m \rightarrow (H^m)^\parallel$ is continuous. Therefore, we know that $T_f : \mathcal{E}_\delta^\infty \rightarrow (\mathcal{E}_\delta^\infty)^\parallel$. With this in mind, it suffices to select

$$V_0^{2\parallel}(\cdot) := - \int_0^{+\infty} (G^{2\parallel} - T_f G^{1\perp})(s, \cdot) ds \in (H^\infty)^\parallel. \quad (\text{III.57})$$

■

• **Proof of Proposition III.5** • The matter here is to show by induction on $K \in \{0, \dots, N+1\}$ the property $\mathcal{H}(N)$ given at the level of line (III.33).

◦ *The condition $\mathcal{H}(0)$ is true.* Noting $V_0^s := {}^t\Phi v_0^s$ and $V_0^f := {}^t\Phi v_0^f$, this preliminary step amounts to the same thing as showing the existence of $V_0^s \in (\mathcal{H}_T^\infty)^{tot}$ and $V_0^f \in (\mathcal{E}_\delta^\infty)^{tot}$ such that

$$\mathcal{A} V_0^s = 0, \quad \mathcal{B} V_0^f = 0, \quad (V_0^s + V_0^f)(0, \cdot) = {}^t\Phi v_0^0(\cdot). \quad (\text{III.58})$$

The above initial condition can be decomposed into

$${}^t(V_0^{s1\perp}, V_0^{s2\perp})(0, \cdot) + {}^t(V_0^{f1\perp}, V_0^{f2\perp})(0, \cdot) = (I - \Pi)v_0^0(\cdot), \quad (\text{III.59})$$

$${}^t(V_0^{s1\parallel}, V_0^{s2\parallel})(0, \cdot) + {}^t(V_0^{f1\parallel}, V_0^{f2\parallel})(0, \cdot) = \Pi v_0^0(\cdot). \quad (\text{III.60})$$

In view of (III.56) and (III.57), we must have $V_0^{f1\parallel}(0, \cdot) \equiv V_0^{f2\parallel}(0, \cdot) \equiv 0$ whatever $V_0^{f\perp}(0, \cdot)$ is. It follows that we can identify $V_0^{s1\parallel}(0, \cdot)$ and $V_0^{s2\parallel}(0, \cdot)$ through (III.60). Now, knowing what is $V_0^{s1\parallel}(0, \cdot)$, formulas (III.50) and (III.52) give access to $V_0^{s1\perp}(0, \cdot)$ and $V_0^{s2\perp}(0, \cdot)$. It remains to use the condition (III.59) in order to further extract $V_0^{f1\perp}(0, \cdot)$ and $V_0^{f2\perp}(0, \cdot)$.

We apply Lemma III.8 and Lemma III.9 in the case of the initial datas $V_0^{s\parallel}(0, \cdot)$ and $V_0^{f\perp}(0, \cdot)$ which have just been computed. Note that, due to the preceding construction, there is no contradiction between the expressions $V_0^{s\perp}(0, \cdot)$ and $V_0^{f\parallel}(0, \cdot)$ thus obtained and the compatibility conditions required at the level of (III.59) and (III.60). By this way, we can recover functions $V_0^s \in (\mathcal{H}_T^\infty)^{tot}$ and $V_0^f \in (\mathcal{E}_\delta^\infty)^{tot}$. Then, to conclude, it suffices to come back to $v_0^s \in \mathcal{H}_T^\infty$ and $v_0^f \in \mathcal{E}_\delta^\infty$ through the action of Φ^{-1} .

◦ *Assume that the condition $\mathcal{H}(K)$ is true for some $K \in \{0, \dots, N\}$.* What must be done here is a repetition of what has been made when $K = 0$, except that we must take source terms into account. Again, we have to combine informations obtained in the proofs of Lemma III.8 and Lemma III.9 together with compatibility conditions on the initial datas. For the sake of completeness, we produce all details below.

Defining $V_{K+1}^s := {}^t\Phi v_{K+1}^s$ and $V_{K+1}^f := {}^t\Phi v_{K+1}^f$, the starting point is the interpretation of (III.29) and (III.30) for the indice $K+1$ into (III.46) and (III.53) and some compatibility restriction issued from (III.31). We have to deal with

$$\mathcal{A} V_{K+1}^s = F_{K+1}, \quad \mathcal{B} V_{K+1}^f = G_{K+1}, \quad (V_{K+1}^s + V_{K+1}^f)(0, \cdot) = {}^t\Phi v_{K+1}^0(\cdot) \quad (\text{III.61})$$

As already noted, the source terms F_{K+1} and G_{K+1} are functions of the V_j^s and V_j^f with $j \leq K$. Due to the induction hypothesis $\mathcal{H}(K)$, these expressions are wellknown. Moreover,

they satisfy $F_{K+1} \in (\mathcal{H}_T^\infty)^{tot}$ and $G_{K+1} \in (\mathcal{E}_\delta^\infty)^{tot}$. The initial condition of (III.61) can be decomposed into

$${}^t(V_{K+1}^{s1\perp}, V_{K+1}^{s2\perp})(0, \cdot) + {}^t(V_{K+1}^{f1\perp}, V_{K+1}^{f2\perp})(0, \cdot) = (I - \Pi)v_{K+1}^0(\cdot), \quad (\text{III.62})$$

$${}^t(V_{K+1}^{s1\parallel}, V_{K+1}^{s2\parallel})(0, \cdot) + {}^t(V_{K+1}^{f1\parallel}, V_{K+1}^{f2\parallel})(0, \cdot) = \Pi v_{K+1}^0(\cdot). \quad (\text{III.63})$$

In view of (III.56) and (III.57), as clearly indicated in the statement of Lemma III.9, the expression $V_{K+1}^{f\parallel}(0, \cdot)$ depends only on G_{K+1} . We can determine $V_{K+1}^{s\parallel}(0, \cdot)$ through (III.63). Knowing what is $V_{K+1}^{s\parallel}(0, \cdot)$, formulas (III.50) and (III.52) give access to $V_{K+1}^{s\perp}(0, \cdot)$. It remains to use the condition (III.62) in order to deduce $V_{K+1}^{f1\perp}(0, \cdot)$ and $V_{K+1}^{f2\perp}(0, \cdot)$. Remark that the initial datas $V_{K+1}^s(0, \cdot)$ and $V_{K+1}^f(0, \cdot)$ thus obtained inherit the expected $\mathcal{H}^\infty(\mathbb{T} \times \mathbb{R})$ smoothness.

Again, we apply Lemmas III.8 and III.9 in the case of the initial datas $V_{K+1}^{s\parallel}(0, \cdot)$ and $V_{K+1}^{f\perp}(0, \cdot)$ which have just been computed. As before, the preceding choices concerning $V_{K+1}^{f\parallel}(0, \cdot)$ and $V_{K+1}^{s\perp}(0, \cdot)$ are sufficient to guarantee (III.63) and (III.62). We find that $V_{K+1}^s \in (\mathcal{H}_T^\infty)^{tot}$ and $V_{K+1}^f \in (\mathcal{E}_\delta^\infty)^{tot}$. To conclude, it suffices to come back to $v_{K+1}^s \in \mathcal{H}_T^\infty$ and $v_{K+1}^f \in \mathcal{E}_\delta^\infty$ through Φ^{-1} .

The induction is finished. ■

As a by-product of the preceding analysis, look at (III.50) and (III.51) in the context of the indice k , we can also retain the following statement.

Corollary III.10 (Linear homogenization). *For all $k \in \{0, \dots, N+1\}$, the expression Πv_k^s can be determined through the following parabolic equation*

$$\partial_t \Pi v_k^s - \left(\mu + \frac{1}{\mu} \Pi((\partial_\theta^{-1} h)^2) \right) \partial_{yy} \Pi v_k^s = S_k^l \quad (\text{III.64})$$

where the source term $S_k^l := {}^t(S_{1,k}^l, S_{2,k}^l)$ depends on the indices j with $j < k$. This fact may be formulated by writing ${}^t(S_{1,k}^l, S_{2,k}^l) = {}^t(f_{1,k}^l, f_{2,k}^l)(v_0^s, \dots, v_{k-1}^s)$.

2.1.2 The case $M \geq 2$

In Subsection 2.1.2, we only assume that $M \geq 2$. It follows that non linear effects are present. In particular, we can no longer seek two approximated solutions v_ε^s and v_ε^f of \mathcal{L}^a and sum them into $v_\varepsilon^s + v_\varepsilon^f$ in order to obtain an expansion satisfying (III.8). We are forced to work with the two time scales t and $\varepsilon^{-2}t$ together. By plugging the expression $v_\varepsilon^s + v_\varepsilon^f$ into \mathcal{L}^a with

the form indicated at the level of (III.4), we can obtain:

$$\begin{aligned}
& \partial_t v_\varepsilon^s(t, \cdot) + \varepsilon^{-1} h \partial_y v_\varepsilon^s(t, \cdot) + \varepsilon^{M-2} (v_\varepsilon^{1s}(t, \cdot) \partial_\theta v_\varepsilon^s(t, \cdot) + \varepsilon v_\varepsilon^{2s}(t, \cdot) \partial_y v_\varepsilon^s(t, \cdot)) \\
& \quad + \varepsilon^{-2} t (0, \partial_\theta h v_\varepsilon^{1s}(t, \cdot)) - \tilde{\mathcal{P}}_\varepsilon v_\varepsilon^s(t, \cdot) \\
& + \varepsilon^{-2} \partial_\tau v_\varepsilon^f(t/\varepsilon^2, \cdot) + \varepsilon^{-1} h \partial_y v_\varepsilon^f(t/\varepsilon^2, \cdot) + \varepsilon^{M-2} v_\varepsilon^{1f}(t/\varepsilon^2, \cdot) \partial_\theta v_\varepsilon^f(t/\varepsilon^2, \cdot) \\
& \quad + \varepsilon^{M-1} v_\varepsilon^{2f}(t/\varepsilon^2, \cdot) \partial_y v_\varepsilon^f(t/\varepsilon^2, \cdot) \\
& \quad + \varepsilon^{-2} t (0, \partial_\theta h v_\varepsilon^{1f}(t/\varepsilon^2, \cdot)) - \tilde{\mathcal{P}}_\varepsilon v_\varepsilon^f(t/\varepsilon^2, \cdot) \\
& \quad + \varepsilon^{M-2} (v_\varepsilon^{1s}(t, \cdot) \partial_\theta v_\varepsilon^f(t/\varepsilon^2, \cdot) + \varepsilon v_\varepsilon^{2s}(t, \cdot) \partial_y v_\varepsilon^f(t/\varepsilon^2, \cdot)) \\
& \quad + \varepsilon^{M-2} v_\varepsilon^{1f}(t/\varepsilon^2, \cdot) \partial_\theta v_\varepsilon^s(t, \cdot) \\
& \quad + \varepsilon^{M-1} v_\varepsilon^{2f}(t/\varepsilon^2, \cdot) \partial_y v_\varepsilon^s(t, \cdot). \tag{III.65}
\end{aligned}$$

Fix any time $T \in \mathbb{R}_+^*$. Define

$$\mathcal{L}^{as}(\varepsilon, v_\varepsilon^s) := \partial_t v_\varepsilon^s + \varepsilon^{-1} h \partial_y v_\varepsilon^s + \varepsilon^{M-2} (v_\varepsilon^{1s} \partial_\theta v_\varepsilon^s + \varepsilon v_\varepsilon^{2s} \partial_y v_\varepsilon^s) + \varepsilon^{-2} t (0, \partial_\theta h v_\varepsilon^{1s}) - \tilde{\mathcal{P}}_\varepsilon v_\varepsilon^s. \tag{III.66}$$

Recall that $v_\varepsilon^f \in \mathcal{E}_\delta^\infty$ is assumed to be exponentially decreasing with respect to $\tau \in \mathbb{R}_+$. Since

$$e^{-\delta t/\varepsilon^2} = O((\varepsilon^2/t)^\infty) = O(\varepsilon^N), \quad \forall (t, N) \in]0, T] \times \mathbb{N},$$

when looking at the Equation (III.65) for times $t \in]0, T]$ with in view a precision of the size $O(\varepsilon^N)$, all terms involving v_ε^f can be neglected. Now, the idea is simply to extend this (relaxed) smallness requirement on the whole interval $[0, T]$. Briefly, we seek v_ε^s so that

$$\mathcal{L}^{as}(\varepsilon, v_\varepsilon^s) = O(\varepsilon^N), \quad t \in [0, T]. \tag{III.67}$$

The Equation (III.67) can be completed with some initial data

$$v_\varepsilon^s(0, \theta, y) = v_\varepsilon^{s0}(\theta, y) = \sum_{k=0}^{N+1} \varepsilon^k v_k^{s0}(\theta, y). \tag{III.68}$$

Clearly, it suffices to specify v_ε^{s0} to determine what is $v_\varepsilon^s(t, \cdot)$ for $t \in [0, T]$, by solving the Cauchy problem (III.67)-(III.68). Now, in order to select v_ε^{s0} conveniently, we have to take into account what happens for small times, in a boundary layer of size ε^2 near $t = 0$. To understand why, just come back to the study of (III.65) for $t \simeq \varepsilon^2$ or $\tau \simeq 1$. Then, the contributions brought by v_ε^f can no more be neglected. Considering (III.65) with the information (III.67) in mind, it seems natural to impose

$$\mathcal{L}^{af}(\varepsilon, v_\varepsilon^f, v_\varepsilon^s) = O(\varepsilon^N), \quad \tau \in [0, 1] \tag{III.69}$$

where we have introduced

$$\begin{aligned}
\mathcal{L}^{af}(\varepsilon, v_\varepsilon^f, v_\varepsilon^s) &:= \varepsilon^{-2} \partial_\tau v_\varepsilon^f(\tau, \cdot) + \varepsilon^{-1} h \partial_y v_\varepsilon^f(\tau, \cdot) - \tilde{\mathcal{P}}_\varepsilon v_\varepsilon^f(\tau, \cdot) \\
& \quad + \varepsilon^{M-2} (v_\varepsilon^{1f}(\tau, \cdot) \partial_\theta v_\varepsilon^f(\tau, \cdot) + \varepsilon v_\varepsilon^{2f}(\tau, \cdot) \partial_y v_\varepsilon^f(\tau, \cdot)) + \varepsilon^{-2t} (0, \partial_\theta h v_\varepsilon^{1f}(\tau, \cdot)) \\
& \quad + \varepsilon^{M-2} (v_\varepsilon^{1s}(\varepsilon^2 \tau, \cdot) \partial_\theta v_\varepsilon^f(\tau, \cdot) + \varepsilon v_\varepsilon^{2s}(\varepsilon^2 \tau, \cdot) \partial_y v_\varepsilon^f(\tau, \cdot)) \\
& \quad + \varepsilon^{M-2} (v_\varepsilon^{1f}(\tau, \cdot) \partial_\theta v_\varepsilon^s(\varepsilon^2 \tau, \cdot) + \varepsilon v_\varepsilon^{2f}(\tau, \cdot) \partial_y v_\varepsilon^s(\varepsilon^2 \tau, \cdot)). \tag{III.70}
\end{aligned}$$

Assume that the data v_ε^{s0} is known. The Cauchy problem (III.67)-(III.68) furnishes $v_\varepsilon^s(t, \cdot)$ for $t \in [0, T]$. In particular, it gives access to all derivatives $(\partial_\tau)^l v_\varepsilon^{*s}(0, \cdot)$ with $l \in \mathbb{N}$. Therefore, we can go further in the analysis by replacing in $\mathcal{L}^{af}(\varepsilon, v_\varepsilon^f, v_\varepsilon^s)$ all the expressions $v_\varepsilon^{*s}(\varepsilon^2 \tau, \cdot)$ by their corresponding Taylor expansions (say up to the order $N-1$) near $t=0$. As long as $\tau \in \mathbb{R}$ is fixed, this operation is justified. From now on, we look at

$$\mathcal{L}^{aft}(\varepsilon, v_\varepsilon^f) = O(\varepsilon^N), \quad \tau \in \mathbb{R}_+ \quad (\text{III.71})$$

where the definition of \mathcal{L}^{aft} is

$$\begin{aligned} \mathcal{L}^{aft}(\varepsilon, v_\varepsilon^f) := & \varepsilon^{-2} \partial_\tau v_\varepsilon(\tau, \cdot) + \varepsilon^{-1} h \partial_y v_\varepsilon(\tau, \cdot) - \tilde{\mathcal{P}}_\varepsilon v_\varepsilon(\tau, \cdot) \\ & + \varepsilon^{M-2} (v_\varepsilon^1(\tau, \cdot) \partial_\theta v_\varepsilon(\tau, \cdot) + \varepsilon v_\varepsilon^2(\tau, \cdot) \partial_y v_\varepsilon(\tau, \cdot)) + \varepsilon^{-2} {}^t(0, \partial_\theta h v_\varepsilon^1(\tau, \cdot)) \\ & + \varepsilon^{M-2} \left(\sum_{l=0}^{N-1} \frac{(\varepsilon^2 \tau)^l}{l!} (\partial_t)^l v_\varepsilon^{1s}(0, \cdot) \partial_\theta v_\varepsilon(\tau, \cdot) + \varepsilon \sum_{l=0}^{N-1} \frac{(\varepsilon^2 \tau)^l}{l!} (\partial_t)^l v_\varepsilon^{2s}(0, \cdot) \partial_y v_\varepsilon(\tau, \cdot) \right) \\ & + \varepsilon^{M-2} \left(v_\varepsilon^1(\tau, \cdot) \sum_{l=0}^{N-1} \frac{(\varepsilon^2 \tau)^l}{l!} (\partial_t)^l \partial_\theta v_\varepsilon^s(0, \cdot) + \varepsilon v_\varepsilon^2(\tau, \cdot) \sum_{l=0}^{N-1} \frac{(\varepsilon^2 \tau)^l}{l!} (\partial_t)^l \partial_y v_\varepsilon^s(0, \cdot) \right). \end{aligned} \quad (\text{III.72})$$

To be coherent with (III.4), we have to impose

$$v_\varepsilon^f(0, \cdot) = v_\varepsilon^{f0}(\cdot) := (v_\varepsilon^a - v_\varepsilon^s)(0, \cdot), \quad (\text{III.73})$$

where $v_\varepsilon^a(0, \cdot)$ and $v_\varepsilon^s(0, \cdot)$ are prescribed as indicated in lines (III.7) and (III.68). Now, we can recover some $v_\varepsilon^f(\tau, \cdot)$ for $\tau \in \mathbb{R}_+$ by solving the Cauchy problem (III.71)-(III.73).

The difficulty comes from the condition $v_\varepsilon^f \in \mathcal{E}_\delta^\infty$. Nothing guarantees that the criterion $v_\varepsilon^f \in \mathcal{E}_\delta^\infty$ can be verified for some well-chosen v_ε^{s0} . To show the existence and the uniqueness of such a data v_ε^{s0} is in fact what matters. The extraction of an adequate function v_ε^{s0} is clarified in the construction described below.

For the sake of brevity, for $k \geq M$, introduce the following notations:

$$\mathcal{J}(M, k) := \{(i, j) \in \{0, \dots, N+1\}^2 ; i+j = k-M\}, \quad (\text{III.74})$$

$$\mathcal{I}(M, k) := \{(i, j, l) \in \{0, \dots, N+1\}^2 \times \{0, \dots, N-1\} ; i+j+2l = k-M\}. \quad (\text{III.75})$$

For $k < M$, we set $\mathcal{J}(M, k) = \emptyset$ and $\mathcal{I}(M, k) = \emptyset$. We also adopt the conventions $v_k^s \equiv v_k^f \equiv 0$ for $k = -3$, $k = -2$ and $k = -1$. Let us now go into the details of the BKW calculus.

The first step is to inject some expansion v_ε^s like in (III.28) into the Equation (III.67). By this way, we can obtain a cascade of equations concerning the unknowns $v_k^s = {}^t(v_{1,k}^s, v_{2,k}^s)$. More

precisely, for $k \in \llbracket 0, N+3 \rrbracket$, we have to consider

$$\begin{aligned} \partial_t v_{1,k-2}^s + h \partial_y v_{1,k-1}^s + \sum_{(i,j) \in \mathcal{J}(M,k)} v_{1,i}^s \partial_\theta v_{1,j}^s + \sum_{(i,j) \in \mathcal{J}(M+1,k)} v_{2,i}^s \partial_y v_{1,j}^s \\ = \mu \partial_{\theta\theta} v_{1,k}^s + \mu \partial_{yy} v_{1,k-2}^s + \lambda \partial_{\theta\theta} v_{1,k-1}^s + \lambda \partial_{\theta y} v_{2,k-2}^s, \end{aligned} \quad (\text{III.76a})$$

$$\begin{aligned} \partial_t v_{2,k-2}^s + h \partial_y v_{2,k-1}^s + \sum_{(i,j) \in \mathcal{J}(M,k)} v_{1,i}^s \partial_\theta v_{2,j}^s + \sum_{(i,j) \in \mathcal{J}(M+1,k)} v_{2,i}^s \partial_y v_{2,j}^s + \partial_\theta h v_{1,k}^s \\ = \mu \partial_{\theta\theta} v_{2,k}^s + \mu \partial_{yy} v_{2,k-2}^s + \lambda \partial_{\theta y} v_{1,k-2}^s + \lambda \partial_{yy} v_{2,k-3}^s. \end{aligned} \quad (\text{III.76b})$$

The next step is to plug some expansion v_ε^f like in (III.28) into the Equation (III.71). By this way, we can obtain a cascade of equations concerning the unknowns $v_k^f = {}^t(v_{1,k}^f, v_{2,k}^f)$. More precisely, for $k \in \llbracket 0, N+1 \rrbracket$, we have to consider

$$\begin{aligned} \partial_\tau v_{1,k}^f + h \partial_y v_{1,k-1}^f + \sum_{(i,j) \in \mathcal{J}(M,k)} v_{1,i}^f \partial_\theta v_{1,j}^f + \sum_{(i,j) \in \mathcal{J}(M+1,k)} v_{2,i}^f \partial_y v_{1,j}^f \\ + \sum_{(i,j,l) \in \mathcal{I}(M,k)} \left(v_{1,i}^f (\partial_t)^l (\partial_\theta v_{1,j}^s)(0) + (\partial_t)^l (v_{1,i}^s)(0) \partial_\theta v_{1,j}^f \right) \frac{\tau^l}{l!} \\ + \sum_{(i,j,l) \in \mathcal{I}(M+1,k)} \left(v_{2,i}^f (\partial_t)^l (\partial_y v_{1,j}^s)(0) + (\partial_t)^l (v_{2,i}^s)(0) \partial_y v_{1,j}^f \right) \frac{\tau^l}{l!} \\ = \mu \partial_{\theta\theta} v_{1,k}^f + \mu \partial_{yy} v_{1,k-2}^f + \lambda \partial_{\theta\theta} v_{1,k-1}^f + \lambda \partial_{\theta y} v_{2,k-2}^f, \end{aligned} \quad (\text{III.77a})$$

$$\begin{aligned} \partial_\tau v_{2,k}^f + h v_{2,k-1}^f + \partial_\theta h v_{1,k}^f + \sum_{(i,j) \in \mathcal{J}(M,k)} v_{1,i}^f \partial_\theta v_{2,j}^f + \sum_{(i,j) \in \mathcal{J}(M+1,k)} v_{2,i}^f \partial_\theta v_{2,j}^f \\ + \sum_{(i,j,l) \in \mathcal{I}(M,k)} \left(v_{1,i}^f (\partial_t)^l (\partial_\theta v_{2,j}^s)(0) + (\partial_t)^l (v_{1,i}^s)(0) \partial_\theta v_{2,j}^f \right) \frac{\tau^l}{l!} \\ + \sum_{(i,j,l) \in \mathcal{I}(M+1,k)} \left(v_{2,i}^f (\partial_t)^l (\partial_y v_{2,j}^s)(0) + (\partial_t)^l (v_{2,i}^s)(0) \partial_y v_{2,j}^f \right) \frac{\tau^l}{l!} \\ = \mu \partial_{\theta\theta} v_{2,k}^f + \mu \partial_{yy} v_{2,k-2}^f + \lambda \partial_{\theta y} v_{1,k-2}^f + \lambda \partial_{yy} v_{2,k-3}^f. \end{aligned} \quad (\text{III.77b})$$

We can associate (III.76) and (III.77) with initial data v_k^{s0} and v_k^{f0} satisfying the restriction (III.31).

Proposition III.11 (Solving (III.76) and (III.77) together with (III.31) and the condition $v_k^f \in \mathcal{E}_\delta^\infty$). *Fix a time $T \in \mathbb{R}_+^*$, a number $\delta \in]0, \mu[$ and, for all $k \in \{0, \dots, N+1\}$, functions $v_k^0 \in H^\infty(\mathbb{T} \times \mathbb{R})$. Then, the conditions (III.76*), (III.77*) and (III.31) have a unique solution such that*

$$(v_k^s, v_k^f) \in \mathcal{H}_T^\infty \times \mathcal{E}_\delta^\infty, \quad \forall k \in \{0, \dots, N+1\}. \quad (\text{III.78})$$

Moreover, the component v_k^s can be identified through the homogenized equation (III.9).

The strategy to show the Proposition III.11 is essentially the same as in the linear case. The proof relies on some induction on the size of N , based on the following hypothesis of induction:

$$\mathcal{HN}(N) : \text{ " The Proposition III.11 is verified up to the integer } N \text{ " .} \quad (\text{III.79})$$

To go from N to $N+1$, we repeat the steps of the subsection 2.1.1. Again, we use a succession of Lemmas which are adaptations of the preceding ones. But now, along the way, we have to take into account the influence of the non linear terms. Before going into the details of the analysis, we give below a brief description of what happens.

- When $M \geq 3$, the construction can be obtained by slightly modifying the linear case.
- When $M = 2$, the non linearity becomes critical and a few arguments must be added. For instance, if we write the equation (III.76a) for $k = 2$, we can notice a Burgers' term

$$\partial_t v_{1,0}^s + h \partial_y v_{1,1}^s + v_{1,0}^s \partial_\theta v_{1,0}^s = \mu \partial_{\theta\theta} v_{1,2}^s + \mu \partial_{yy} v_{1,0}^s + \lambda \partial_{\theta\theta} v_{1,1}^s + \lambda \partial_{\theta y} v_{2,0}^s \quad (\text{III.80})$$

and also two contributions $\partial_{\theta\theta} v_{1,2}^s$ and $\partial_{\theta\theta} v_{1,1}^s$ to be calculated in function of $v_{1,0}^s$, with apparently a non linear dependence with respect to $v_{1,0}^s$.

- When $M = 2$ again, another effect of the non linear interactions is the apparition at the level of what replaces the homogenized Equation (III.37), see (III.82) below together with (III.86), of a new source term $S_k^{nl//}$ which can depend on v_k^s .
- However, there are *transparency phenomena* at work which come from the initialization procedure. Indeed, knowing that $v_k^s \equiv 0$ for $k \in \{-3, -2, -1\}$, the Equation (III.76) in the case $k = 0$ and $M \geq 2$ reduces to $\mu \partial_{\theta\theta} v_0^s = 0$. In other words, we have to impose

$$v_{1,0}^{s\perp} \equiv (I - \Pi) v_{1,0}^s \equiv 0. \quad (\text{III.81})$$

It follows that $v_{1,0}^s \partial_\theta v_{1,0}^s \equiv 0$. In the same way, all other apparent non linear contributions will disappear. Therefore, the remaining term $\Pi v_{1,0}^s$ can be determined apart without seeing any non linear effect.

• **Interpretation of the system (III.76)** • The first step (Lemma III.12 below) extends to the framework of system (III.76) what has been done in the proof of Lemma III.7. The new thing is the production of additional source terms $S_k^{nl//}$ with explicit formulas. The further step in this paragraph (Lemma III.13) describes precisely how the $S_k^{nl//}$ depend on the v_j^s . For this occasion, the two cases $M \geq 3$ and $M = 2$ must be distinguished.

Lemma III.12 (Non-linear homogenization). *Assume that the functions v_k^s with $k \in \{0, \dots, N+3\}$ are solutions of the system (III.76). Then, for all $k \in \{0, \dots, N+1\}$, the part $\Pi v_k^s = {}^t(\Pi v_{1,k}^s, \Pi v_{2,k}^s)$ is a solution of*

$$\begin{cases} \partial_t \Pi v_{1,k}^s - \left(\mu + \frac{1}{\mu} \Pi((\partial_\theta^{-1} h)^2) \right) \partial_{yy} \Pi v_{1,k}^s = S_{1,k}^{l//} + S_{1,k}^{nl//} + P_1^{l\perp} (I - \Pi) v_{1,k}^s, \\ \partial_t \Pi v_{2,k}^s - \left(\mu + \frac{1}{\mu} \Pi((\partial_\theta^{-1} h)^2) \right) \partial_{yy} \Pi v_{2,k}^s = S_{2,k}^{l//} + S_{2,k}^{nl//} + P_2^{l\perp} (I - \Pi) v_{1,k}^s \\ \quad + P_2^{l//} \Pi v_{1,k}^s + Q_2^{l\perp} (I - \Pi) v_{2,k}^s. \end{cases} \quad (\text{III.82})$$

In the above system (III.82), the four operators $P_1^{l\perp}$, $P_2^{l\perp}$, $P_2^{l//}$ and $Q_2^{l\perp}$, as well as the source term $S_k^{l//} = {}^t(S_{1,k}^{l//}, S_{2,k}^{l//})$, are those defined along lines (III.38)–(III.45). On the other hand, the new contribution $S_k^{nl//} = {}^t(S_{1,k}^{nl//}, S_{2,k}^{nl//})$ is given by (III.83)–(III.84).

Proof of lemma III.12. The matter is to identify the contributions brought by the non linear terms. Let us project (III.76) according to $\mathcal{V}^//$. Since $\mathcal{J}(M+1, k+2) \equiv \mathcal{J}(M, k+1)$, this yields

$$\begin{aligned} & \partial_t \Pi v_{1,k}^s + \Pi(h \partial_y v_{1,k+1}^s) + \Pi \left(\sum_{(i,j) \in \mathcal{J}(M,k+2)} v_{1,i}^s \partial_\theta v_{1,j}^s \right) \\ & + \Pi \left(\sum_{(i,j) \in \mathcal{J}(M,k+1)} v_{2,i}^s \partial_y v_{1,j}^s \right) = \mu \partial_{yy} \Pi v_{1,k}^s, \\ & \partial_t \Pi v_{2,k}^s + \Pi(h \partial_y v_{2,k+1}^s) + \Pi \left(\sum_{(i,j) \in \mathcal{J}(M,k+2)} v_{1,i}^s \partial_\theta v_{2,j}^s \right) \\ & + \Pi \left(\sum_{(i,j) \in \mathcal{J}(M,k+1)} v_{2,i}^s \partial_y v_{2,j}^s \right) + \Pi(\partial_\theta h v_{1,k+2}^s) = \mu \partial_{yy} \Pi v_{2,k}^s + \lambda \partial_{yy} \Pi v_{2,k-1}^s. \end{aligned}$$

In what follows, we will use the system (III.76) and many integrations by parts in order to interpret $\Pi(h \partial_y v_{1,k+1}^s)$, $\Pi(h \partial_y v_{2,k+1}^s)$ and $\Pi(\partial_\theta h v_{1,k+2}^s)$. The goal is to show that these quantities can be expressed in terms of the v_j^s with $j \in \{0, \dots, k\}$.

◦ Study of $\Pi(h \partial_y v_{1,k+1}^s)$. Exploiting (III.76a) with $k+1$ in place of k , we can deduce

$$\begin{aligned} \Pi(h \partial_y v_{1,k+1}^s) &= \Pi(\partial_{\theta\theta}^{-2}(h) \partial_y \partial_{\theta\theta} v_{1,k+1}^s) \\ &= \frac{1}{\mu} \Pi(\partial_{\theta\theta}^{-2}(h) \partial_y (\partial_t v_{1,k-1}^s - \mu \partial_{yy} v_{1,k-1}^s - \lambda \partial_{\theta y} v_{2,k-1}^s)) \\ &+ \frac{1}{\mu} \Pi \left(\partial_{\theta\theta}^{-2}(h) \partial_y \left(\sum_{(i,j) \in \mathcal{J}(M,k+1)} v_{1,i}^s \partial_\theta v_{1,j}^s + \sum_{(i,j) \in \mathcal{J}(M,k)} v_{2,i}^s \partial_y v_{1,j}^s \right) \right) \\ &+ \frac{1}{\mu} \Pi(h \partial_{\theta\theta}^{-2}(h) \partial_{yy} v_{1,k}^s) - \frac{\lambda}{\mu} \Pi(\partial_{\theta\theta}^{-2}(h) \partial_{y\theta\theta} v_{1,k}^s). \end{aligned}$$

Recalling (III.34) and (III.35) together with the conventions (III.38) and (III.39), we have to deal with

$$\begin{aligned} \Pi(h \partial_y v_{1,k+1}^s) &= -\frac{1}{\mu} \Pi((\partial_\theta^{-1} h)^2) \partial_{yy} \Pi v_{1,k}^s - S_{1,k}^{l//} - P_1^{l\perp}(I - \Pi) v_{1,k}^s \\ &+ \frac{1}{\mu} \Pi \left(\partial_{\theta\theta}^{-2}(h) \partial_y \left(\sum_{(i,j) \in \mathcal{J}(M,k+1)} v_{1,i}^s \partial_\theta v_{1,j}^s + \sum_{(i,j) \in \mathcal{J}(M,k)} v_{2,i}^s \partial_y v_{1,j}^s \right) \right). \end{aligned}$$

◦ Study of $\Pi(h \partial_y v_{2,k+1}^s)$. Exploiting (III.76b) with $k+1$ in place of k , we can derive

$$\begin{aligned} \Pi(h \partial_y v_{2,k+1}^s) &= \Pi(\partial_{\theta\theta}^{-2}(h) \partial_y \partial_{\theta\theta} v_{2,k+1}^s) \\ &= \frac{1}{\mu} \Pi(\partial_{\theta\theta}^{-2}(h) \partial_y (\partial_t v_{2,k-1}^s - \mu \partial_{yy} v_{2,k-1}^s - \lambda \partial_{\theta y} v_{1,k-1}^s - \lambda \partial_{yy} v_{2,k-2}^s)) \\ &\quad + \frac{1}{\mu} \Pi \left(\partial_{\theta\theta}^{-2}(h) \partial_y \left(\sum_{(i,j) \in \mathcal{J}(M,k+1)} v_{1,i}^s \partial_{\theta} v_{2,j}^s + \sum_{(i,j) \in \mathcal{J}(M,k)} v_{2,i}^s \partial_y v_{2,j}^s \right) \right) \\ &\quad + \frac{1}{\mu} \Pi(\partial_{\theta\theta}^{-2}(h) \partial_y (h \partial_y v_{2,k}^s + \partial_{\theta} h v_{1,k+1}^s)). \end{aligned}$$

We come back to the equation (III.76a) in order to change the last term in this sum. This yields

$$\begin{aligned} \frac{1}{\mu} \Pi(\partial_{\theta\theta}^{-2}(h) \partial_y (\partial_{\theta} h v_{1,k+1}^s)) &= \frac{1}{\mu} \Pi(\partial_{\theta} h \partial_{\theta\theta}^{-2}(h) \partial_y (I - \Pi) v_{1,k+1}^s) \\ &= \frac{1}{\mu^2} \Pi(\partial_{\theta} h \partial_{\theta\theta}^{-2}(h) \partial_y \partial_{\theta\theta}^{-2}(I - \Pi) (\partial_t v_{1,k-1}^s - \mu \partial_{yy} v_{1,k-1}^s - \lambda \partial_{\theta y} v_{2,k-1}^s)) \\ &\quad + \frac{1}{\mu^2} \Pi(\partial_{\theta} h \partial_{\theta\theta}^{-2}(h) \partial_y \partial_{\theta\theta}^{-2}(I - \Pi) (h \partial_y v_{1,k}^s - \lambda \partial_{\theta\theta} v_{1,k}^s)) \\ &\quad + \frac{1}{\mu^2} \Pi \left(\partial_{\theta} h \partial_{\theta\theta}^{-2}(h) \partial_y \partial_{\theta\theta}^{-2}(I - \Pi) \left(\sum_{(i,j) \in \mathcal{J}(M,k+1)} v_{1,i}^s \partial_{\theta} v_{1,j}^s + \sum_{(i,j) \in \mathcal{J}(M,k)} v_{2,i}^s \partial_y v_{1,j}^s \right) \right). \end{aligned}$$

Using notations (III.40), (III.41) and (III.42), this amounts to the same thing as

$$\begin{aligned} \Pi(h \partial_y v_{2,k+1}^s) &= -\frac{1}{\mu} \Pi((\partial_{\theta}^{-2} h)^2) \partial_{yy} \Pi v_{2,k}^s \\ &\quad - S_{2,k}^{l1//} - P_2^{l1\perp} (I - \Pi) v_{1,k}^s - P_2^{l1//} \Pi v_{1,k}^s - Q_2^{l1\perp} (I - \Pi) v_{2,k}^s \\ &\quad + \frac{1}{\mu} \Pi \left(\partial_{\theta\theta}^{-2}(h) \partial_y \left(\sum_{(i,j) \in \mathcal{J}(M,k+1)} v_{1,i}^s \partial_{\theta} v_{1,j}^s + \sum_{(i,j) \in \mathcal{J}(M,k)} v_{2,i}^s \partial_y v_{1,j}^s \right) \right) \\ &\quad + \frac{1}{\mu^2} \Pi \left(\partial_{\theta} h \partial_{\theta\theta}^{-2}(h) \partial_y \partial_{\theta\theta}^{-2}(I - \Pi) \left(\sum_{(i,j) \in \mathcal{J}(M,k+1)} v_{1,i}^s \partial_{\theta} v_{1,j}^s + \sum_{(i,j) \in \mathcal{J}(M,k)} v_{2,i}^s \partial_y v_{1,j}^s \right) \right). \end{aligned}$$

◦ It remains to compute $\Pi(\partial_{\theta} h v_{1,k+2}^s)$. This is again (III.76a) with this time $k+2$ in place

of k .

$$\begin{aligned}\Pi(\partial_\theta h v_{1,k+2}^s) &= \Pi(\partial_\theta^{-1}(h) \partial_{\theta\theta} v_{1,k+2}^s) \\ &= \frac{1}{\mu} \Pi(\partial_\theta^{-1}(h) (\partial_t v_{1,k}^s - \mu \partial_{yy} v_{1,k}^s - \lambda \partial_{\theta y} (I - \Pi) v_{2,k}^s)) \\ &\quad + \frac{1}{\mu} \Pi(\partial_\theta^{-1}(h) (h \partial_y v_{1,k+1}^s - \lambda \partial_{\theta\theta} v_{1,k+1}^s)) \\ &\quad + \frac{1}{\mu} \Pi \left(\partial_\theta^{-1}(h) \left(\sum_{(i,j) \in \mathcal{J}(M,k+2)} v_{1,i}^s \partial_\theta v_{1,j}^s + \sum_{(i,j) \in \mathcal{J}(M,k+1)} v_{2,i}^s \partial_y v_{1,j}^s \right) \right).\end{aligned}$$

We want to remove the presence of $v_{1,k+1}^s$. The relation $\Pi(h \partial_\theta^{-1} h) = 0$ allows to write

$$\Pi(\partial_\theta^{-1}(h) (h \partial_y v_{1,k+1}^s - \lambda \partial_{\theta\theta} v_{1,k+1}^s)) = \Pi(\partial_\theta^{-1}(h) (h \partial_y - \lambda \partial_{\theta\theta}) \partial_{\theta\theta}^{-2} (I - \Pi) \partial_{\theta\theta} v_{1,k+1}^s).$$

The part $\partial_{\theta\theta} v_{1,k+1}^s$ can be extracted from the equation (III.76a) with $k+1$ in place of k . We find

$$\begin{aligned}\Pi(\partial_\theta h v_{1,k+2}^s) &= \frac{1}{\mu} \Pi(\partial_\theta^{-1}(h) (\partial_t v_{1,k}^s - \mu \partial_{yy} v_{1,k}^s - \lambda \partial_{\theta y} (I - \Pi) v_{2,k}^s)) \\ &\quad + \frac{1}{\mu^2} \Pi(\partial_\theta^{-1}(h) (h \partial_y - \lambda \partial_{\theta\theta}) \partial_{\theta\theta}^{-2} (I - \Pi) (\partial_t v_{1,k-1}^s - \mu \partial_{yy} v_{1,k-1}^s - \lambda \partial_{\theta y} v_{2,k-1}^s)) \\ &\quad + \frac{1}{\mu^2} \Pi(\partial_\theta^{-1}(h) (h \partial_y - \lambda \partial_{\theta\theta}) \partial_{\theta\theta}^{-2} (I - \Pi) (h \partial_y v_{1,k}^s - \lambda \partial_{\theta\theta} v_{1,k}^s)) \\ &\quad + \frac{1}{\mu^2} \Pi \left(\partial_\theta^{-1}(h) (h \partial_y - \lambda \partial_{\theta\theta}) \partial_{\theta\theta}^{-1} (I - \Pi) \left(\sum_{(i,j) \in \mathcal{J}(M,k+1)} v_{1,i}^s \partial_\theta v_{1,j}^s + \sum_{(i,j) \in \mathcal{J}(M,k)} v_{2,i}^s \partial_y v_{1,j}^s \right) \right) \\ &\quad + \frac{1}{\mu} \Pi \left(\partial_\theta^{-1}(h) \left(\sum_{(i,j) \in \mathcal{J}(M,k+2)} v_{1,i}^s \partial_\theta v_{1,j}^s + \sum_{(i,j) \in \mathcal{J}(M,k+1)} v_{2,i}^s \partial_y v_{1,j}^s \right) \right).\end{aligned}$$

With the notations (III.43)-(III.44)-(III.45), this is

$$\begin{aligned}\Pi(\partial_\theta h v_{1,k+2}^s) &= -S_{2,k}^{l2//} - P_2^{l2\perp} (I - \Pi) v_{1,k}^s - P_2^{l2//} \Pi v_{1,k}^s - Q_2^{l2\perp} (I - \Pi) v_{2,k}^s \\ &\quad + \frac{1}{\mu^2} \Pi \left(\partial_\theta^{-1}(h) (h \partial_y - \lambda \partial_{\theta\theta}) \partial_{\theta\theta}^{-2} (I - \Pi) \left(\sum_{(i,j) \in \mathcal{J}(M,k+1)} v_{1,i}^s \partial_\theta v_{1,j}^s + \sum_{(i,j) \in \mathcal{J}(M,k)} v_{2,i}^s \partial_y v_{1,j}^s \right) \right) \\ &\quad + \frac{1}{\mu} \Pi \left(\partial_\theta^{-1}(h) \left(\sum_{(i,j) \in \mathcal{J}(M,k+2)} v_{1,i}^s \partial_\theta v_{1,j}^s + \sum_{(i,j) \in \mathcal{J}(M,k+1)} v_{2,i}^s \partial_y v_{1,j}^s \right) \right).\end{aligned}$$

◦ Conclusion. Combining all informations together, we finally obtain (III.82) with

$$\begin{aligned}S_{1,k}^{nl//} &:= -\frac{1}{\mu} \Pi \left(\partial_{\theta\theta}^{-2}(h) \partial_y \left(\sum_{(i,j) \in \mathcal{J}(M,k+1)} v_{1,i}^s \partial_\theta v_{1,j}^s + \sum_{(i,j) \in \mathcal{J}(M,k)} v_{2,i}^s \partial_y v_{1,j}^s \right) \right) \\ &\quad - \Pi \left(\sum_{(i,j) \in \mathcal{J}(M,k+2)} v_{1,i}^s \partial_\theta v_{1,j}^s + \sum_{(i,j) \in \mathcal{J}(M,k+1)} v_{2,i}^s \partial_y v_{1,j}^s \right),\end{aligned}\tag{III.83}$$

and

$$\begin{aligned}
S_{2,k}^{nl//} = & -\frac{1}{\mu} \Pi \left(\partial_{\theta\theta}^{-2}(h) \partial_y \left(\sum_{(i,j) \in \mathcal{J}(M,k+1)} v_{1,i}^s \partial_{\theta} v_{1,j}^s + \sum_{(i,j) \in \mathcal{J}(M,k)} v_{2,i}^s \partial_y v_{1,j}^s \right) \right) \\
& - \frac{1}{\mu^2} \Pi \left(\partial_{\theta} h \partial_{\theta\theta}^{-2}(h) \partial_y \partial_{\theta\theta}^{-2}(I - \Pi) \left(\sum_{(i,j) \in \mathcal{J}(M,k+1)} v_{1,i}^s \partial_{\theta} v_{1,j}^s + \sum_{(i,j) \in \mathcal{J}(M,k)} v_{2,i}^s \partial_y v_{1,j}^s \right) \right) \\
& - \frac{1}{\mu^2} \Pi \left(\partial_{\theta}^{-1}(h) (h \partial_y - \lambda \partial_{\theta\theta}) \partial_{\theta\theta}^{-2}(I - \Pi) \left(\sum_{(i,j) \in \mathcal{J}(M,k+1)} v_{1,i}^s \partial_{\theta} v_{1,j}^s + \sum_{(i,j) \in \mathcal{J}(M,k)} v_{2,i}^s \partial_y v_{1,j}^s \right) \right) \\
& - \frac{1}{\mu} \Pi \left(\partial_{\theta}^{-1}(h) \left(\sum_{(i,j) \in \mathcal{J}(M,k+2)} v_{1,i}^s \partial_{\theta} v_{1,j}^s + \sum_{(i,j) \in \mathcal{J}(M,k+1)} v_{2,i}^s \partial_y v_{1,j}^s \right) \right) \\
& - \Pi \left(\sum_{(i,j) \in \mathcal{J}(M,k+2)} v_{1,i}^s \partial_{\theta} v_{2,j}^s + \sum_{(i,j) \in \mathcal{J}(M,k+1)} v_{2,i}^s \partial_y v_{2,j}^s \right). \tag{III.84}
\end{aligned}$$

■

The definition of $S_k^{nl//}$ involves the sets $\mathcal{J}(M, k)$, $\mathcal{J}(M, k+1)$ and $\mathcal{J}(M, k+2)$. Note that

$$[M \geq 3, \tilde{k} \in \{k, k+1, k+2\}, (i, j) \in \mathcal{J}(M, \tilde{k})] \implies i \leq k-1 \text{ and } j \leq k-1. \tag{III.85}$$

Thus, the presence of v_k^s inside $S_k^{nl//}$ is not allowed as long as $M \geq 3$. On the contrary, when $M = 2$, it becomes effective. This remark can be formalized through the following statement.

Lemma III.13 (Refined description of the source term of (III.82) in the non linear case).

i) When $M \geq 3$, the expressions $S_k^{l//}$ and $S_k^{nl//}$ only depend on the v_j^s with $j \in \{0, \dots, k-1\}$. More precisely, we can retain that

$$S_k^{l//} = f_k^{l//}(v_0^s, \dots, v_{k-1}^s), \quad S_k^{nl//} = f_k^{qnl//}(v_0^s, \dots, v_{k-1}^s)$$

where $f_k^{l//} = {}^t(f_{1,k}^{l//}, f_{2,k}^{l//})$ and $f_k^{qnl//} = {}^t(f_{1,k}^{qnl//}, f_{2,k}^{qnl//})$ are respectively homogeneous linear and quadratic functions of their arguments.

ii) When $M = 2$, the expressions $S_k^{nl//}$ only depend on the v_j^s with $j \in \{0, \dots, k\}$. The influence of v_k^s can be precised through a decomposition of the form

$$S_k^{nl//} = f_k^{qnl//}(v_0^s, \dots, v_{k-1}^s) + SP(v_0^s) v_k^s, \quad f_k^{qnl//} = (f_{1,k}^{qnl//}, f_{2,k}^{qnl//}).$$

Again $f_k^{qnl//}$ is a quadratic function of its arguments whereas $SP(v_0^s) := {}^t(SP_1(v_0^s), SP_2(v_0^s))$ is the linear differential operator defined according to

$$\begin{aligned}
SP_1(v_0^s) v_k^s &:= -\Pi(v_{1,k}^s \partial_{\theta} v_{1,0}^s + v_{1,0}^s \partial_{\theta} v_{1,k}^s), \\
SP_2(v_0^s) v_k^s &:= -\Pi(v_{1,0}^s \partial_{\theta} v_{2,k}^s + v_{1,k}^s \partial_{\theta} v_{2,0}^s) - \frac{1}{\mu} \Pi(\partial_{\theta}^{-1}(h) (v_{1,0}^s \partial_{\theta} v_{1,k}^s + v_{1,k}^s \partial_{\theta} v_{1,0}^s)).
\end{aligned} \tag{III.86}$$

From now on, we note simply $f_k^{nl//} := f_k^{l//} + f_k^{qnl//}$.

Proof of lemma III.13. As already mentioned, the statement *i)* is a direct consequence of (III.85). The *linear* aspect of $f_k^{l//}$ is a consequence of the formulas obtained in the Subsection 2.1.1. On the other hand, the *quadratic* aspect of $f_k^{qnl//}$ is obvious in view of (III.83) and (III.84).

It remains to consider the situation *ii)* where $M = 2$. Note that $\mathcal{J}(2, k+2) = \mathcal{J}(0, k)$. In view of (III.85), we have to concentrate on the contribution

$$\left(\begin{array}{c} -\Pi \left(\sum_{(i,j) \in \mathcal{J}(0,k)} v_{1,i}^s \partial_\theta v_{1,j}^s \right) \\ -\frac{1}{\mu} \Pi \left(\partial_\theta^{-1}(h) \left(\sum_{(i,j) \in \mathcal{J}(0,k)} v_{1,i}^s \partial_\theta v_{1,j}^s \right) \right) - \Pi \left(\sum_{(i,j) \in \mathcal{J}(0,k)} v_{1,i}^s \partial_\theta v_{2,j}^s \right) \end{array} \right).$$

In the sums above, only the extremal indices $(i, j) = (k, 0)$ and $(i, j) = (0, k)$ give a contribution to include in $SP(v_0^s)$, leading to (III.86). ■

• **Analysis of the system (III.76)** • Introduce $f_k^{nl\perp} := f_k^{l\perp} + f_k^{qnl\perp}$ where $f_k^{l\perp}$ is defined at the level of (III.47)-(III.48) whereas $f_k^{qnl\perp} = {}^t(f_{1,k}^{qnl\perp}, f_{2,k}^{qnl\perp})$ is given by

$$f_{p,k}^{qnl\perp} := (I - \Pi) \left(\sum_{(i,j) \in \mathcal{J}(M,k)} v_{1,i}^s \partial_\theta v_{p,j}^s + \sum_{(i,j) \in \mathcal{J}(M+1,k)} v_{2,i}^s \partial_y v_{p,j}^s \right), \quad p \in \{1, 2\}. \quad (\text{III.87})$$

Lemma III.14 (Description of $f_k^{nl\perp}$). *The functions $f_k^{l\perp}$ and $f_k^{qnl\perp}$ only depend on the v_j^s with $j \in \{0, \dots, k-1\}$. They are respectively homogeneous linear and homogeneous quadratic functions of their arguments $(v_0^s, \dots, v_{k-1}^s)$.*

Proof of lemma III.14. Just look at (III.47)-(III.48)-(III.87) together with (III.85). ■

The guiding principle to study the system (III.76) is the same as in the subsection 2.1.1. Still, to go further, we have to distinguish the general case $M \geq 3$ from the critical case $M = 2$.

◦ When $M \geq 3$, the expression $V_k^s := {}^t\Phi v_k^s$ turns to be a solution of

$$\mathcal{A} V_k^s = f_k^{nl} := {}^t(f_{1,k}^{nl\perp}, f_{1,k}^{nl//}, f_{2,k}^{nl\perp}, f_{2,k}^{nl//}). \quad (\text{III.88})$$

According to the Lemmas III.7 and III.14, the above right hand term f_k^{nl} can be viewed as a source term. The discussion when solving (III.88) is clearly as in Lemma III.8.

◦ The case $M = 2$. Recall that we can exploit the condition (III.81). This information is essential. It induces many simplifications when computing $SP(v_0^s)$. We find that $SP_1(v_0^s) \equiv 0$ whereas $SP_2(v_0^s)$ can be reduced to the following linear (non differential) operator

$$SP_2(v_0^s) v_k^s = -\Pi(v_{1,k}^{s\perp} \partial_\theta v_{2,0}^s) + \frac{1}{\mu} v_{1,0}^{s//} \Pi(h v_{1,k}^{s\perp}). \quad (\text{III.89})$$

At first sight, the expression $SP_2(v_0^s)v_0^s$ depends in a non linear way on v_0^s . However, we can again exploit the condition (III.81) (which says that $v_{1,0}^{s\perp} \equiv 0$) and then apply (III.89) with $k = 0$ in order to obtain further cancellations. It remains

$$SP(v_0^s)v_0^s \equiv 0. \quad (\text{III.90})$$

From now on, since there is no more ambiguity, we can omit to signal that $SP(v_0^s)$ depends on v_0^s , and in fact only on $v_{1,0}^{s\perp}$. We will most often note $SP(v_0^s) \equiv SP(v_0^{s\perp}) \equiv SP = {}^t(SP_1, SP_2)$. Since $SP_2 \not\equiv 0$, the formulation (III.46) must be changed. This time, we have to deal with

$$\tilde{\mathcal{A}}V_k^s = {}^t(f_{1,k}^{nl\perp}, f_{1,k}^{nl\parallel}, f_{2,k}^{nl\perp}, f_{2,k}^{nl\parallel}), \quad V_k^s := {}^t\Phi v_k^s \quad (\text{III.91})$$

where

$$\tilde{\mathcal{A}} := \begin{pmatrix} \mu\partial_{\theta\theta} & 0 & 0 & 0 \\ -P_1^{l\perp} & P_y & 0 & 0 \\ -T_s & -T_s & \mu\partial_{\theta\theta} & 0 \\ -P_2^{l\perp} - SP_2 & -P_2^{l\perp} & -Q_2^{l\perp} & P_y \end{pmatrix}. \quad (\text{III.92})$$

The similarities between \mathcal{A} and $\tilde{\mathcal{A}}$ are obvious. These two matrix valued operators have both a triangular structure. The difference, when passing from \mathcal{A} to $\tilde{\mathcal{A}}$, concerns only the perturbation in the bottom-left position (4, 1). This particularity plays a crucial part in the discussion below.

Lemma III.15 (Solving the system (III.91) that is the system (III.82) in the case $M = 2$). *We assume that $M = 2$ and that the condition (III.81) is verified. Select a function $V_0^{\parallel} = {}^t(V_0^{1\parallel}, V_0^{2\parallel}) \in H^\infty(\mathbb{R})^2$ and a source term $F = {}^t(F^{1\perp}, F^{1\parallel}, F^{2\perp}, F^{2\parallel}) \in (\mathcal{H}_T^\infty)^{tot}$. Then, for all $T \in \mathbb{R}_+^*$, the problem*

$$\{ \tilde{\mathcal{A}}V = F, \quad V = {}^t(V^{1\perp}, V^{1\parallel}, V^{2\perp}, V^{2\parallel}), \quad {}^t(V^{1\parallel}, V^{2\parallel})|_{t=0} = {}^t(V_0^{1\parallel}, V_0^{2\parallel}) \} \quad (\text{III.93})$$

has a unique solution V in $(\mathcal{H}_T^\infty)^{tot}$.

Proof of lemma III.15. As in Lemma III.8, the successive components of V can be determined by arguing line after line. The only change is at the level of the fourth line where the supplementary source term $SP_2 V^{1\perp}$ must be incorporated. ■

• **Analysis of the system (III.77)** • Introduce $V_k^f := {}^t\Phi v_k^f$. From (III.77), we can extract that V_k^f must be a solution of

$$\mathcal{B}V_k^f = {}^t(g_{1,k}^{nl\perp}, g_{1,k}^{nl\parallel}, g_{2,k}^{nl\perp}, g_{2,k}^{nl\parallel}), \quad g_k^{nl} := g_k^l + g_k^{qnl}. \quad (\text{III.94})$$

The definitions of \mathcal{B} and g_k^l are the same as in (III.53) and (III.54). On the other hand, we

find that the extra source term $g_k^{qnl} = {}^t(g_{1,k}^{qnl}, g_{2,k}^{qnl})$ is given by

$$\begin{aligned}
g_{1,k}^{qnl} &:= \sum_{(i,j) \in \mathcal{J}(M,k)} v_{1,i}^f \partial_\theta v_{1,j}^f + \sum_{(i,j) \in \mathcal{J}(M,k-1)} v_{2,i}^f \partial_y v_{1,j}^f \\
&+ \sum_{(i,j,l) \in \mathcal{I}(M,k)} \left(v_{1,i}^f (\partial_t)^l (\partial_\theta v_{1,j}^s)(0) + (\partial_t)^l (v_{1,i}^s)(0) \partial_\theta v_{1,j}^f \right) \frac{\tau^l}{l!} \\
&+ \sum_{(i,j,l) \in \mathcal{I}(M,k-1)} \left(v_{2,i}^f (\partial_t)^l (\partial_y v_{1,j}^s)(0) + (\partial_t)^l (v_{2,i}^s)(0) \partial_y v_{1,j}^f \right) \frac{\tau^l}{l!}, \\
g_{2,k}^{qnl} &:= \sum_{(i,j) \in \mathcal{J}(M,k)} v_{1,i}^f \partial_\theta v_{2,j}^f + \sum_{(i,j) \in \mathcal{J}(M,k-1)} v_{2,i}^f \partial_\theta v_{2,j}^f \\
&+ \sum_{(i,j,l) \in \mathcal{I}(M,k)} \left(v_{1,i}^f (\partial_t)^l (\partial_\theta v_{2,j}^s)(0) + (\partial_t)^l (v_{1,i}^s)(0) \partial_\theta v_{2,j}^f \right) \frac{\tau^l}{l!} \\
&+ \sum_{(i,j,l) \in \mathcal{I}(M,k-1)} \left(v_{2,i}^f (\partial_t)^l (\partial_y v_{2,j}^s)(0) + (\partial_t)^l (v_{2,i}^s)(0) \partial_y v_{2,j}^f \right) \frac{\tau^l}{l!}.
\end{aligned}$$

Lemma III.16. *Assume that $M \geq 2$. The functions g_k^l and g_k^{qnl} depend only on the v_j^f and the $\partial_t^l v_j^s(0, \cdot)$ where $j \in \{0, \dots, k-1\}$ and $l \in \{0, \dots, \lfloor \frac{k}{2} \rfloor\}$. They are respectively homogeneous linear and homogeneous quadratic functions of their arguments.*

Proof of lemma III.16. It suffices to examine the various terms appearing in the sums involved by the definition of g_k^{qnl} .

- The sums based on the symbol \mathcal{J} can be dealt by observing that

$$[M \geq 2, \tilde{k} \in \{k-1, k\}, (i, j) \in \mathcal{J}(M, \tilde{k})] \implies i \leq k-1 \text{ and } j \leq k-1. \quad (\text{III.95})$$

- The sums involving the symbol \mathcal{I} are of the form $\mathcal{I}(M, k)$ or $\mathcal{I}(M, k-1)$. Coming back to the definition (III.75), we can easily infer that

$$[M \geq 2, \tilde{k} \in \{k-1, k\}, (i, j, l) \in \mathcal{I}(M, \tilde{k})] \implies i \leq k-1 \text{ and } j \leq k-1 \quad (\text{III.96})$$

as well as $l \leq \lfloor \frac{k}{2} \rfloor$. ■

Due to Lemma III.16, the expression g_k^{nl} can be viewed as a source term. Thus, the discussion about (III.94) can be done in the same context as in Lemma III.9.

• **Proof of Proposition III.11** • The matter is to show by induction on $K \in \{0, \dots, N+1\}$ that the property given at the level of line (III.124) is verified. The induction has already been initiated. We have seen that the analysis of (III.76) in the case $k = 0$ yields directly (III.81). Now, knowing that the condition (III.81) is verified, we can interpret all equations as indicated just before. It means that the discussion is very similar to what has been done in subsection 2.1.1, and we can be satisfied with only giving indications of proof.

◦ *Verification of $\mathcal{HN}(0)$.* By convention, we start with $v_k^s \equiv 0$ and $v_k^f \equiv 0$ for $k \in \{-3, -2, -1\}$. Applying Lemmas III.13, III.14 and III.16 with $k = 0$ and exploiting the given (linear or quadratic) homogeneity properties, we find that $f_0^{nl} \equiv 0$ and $g_0^{nl} \equiv 0$.

Recall that $V_0^s := {}^t\Phi v_0^s$ and $V_0^f := {}^t\Phi v_0^f$. The matter here is to show the existence of functions $V_0^s \in (\mathcal{H}_T^\infty)^{tot}$ and $V_0^f \in (\mathcal{E}_\delta^\infty)^{tot}$ such that:

- For $M \geq 3$:

$$\mathcal{A}V_0^s = 0, \quad \mathcal{B}V_0^f = 0, \quad (V_0^s + V_0^f)(0, \cdot) = {}^t\Phi v_0^0(\cdot). \quad (\text{III.97})$$

- For $M = 2$:

$$\tilde{\mathcal{A}}V_0^s = 0, \quad \mathcal{B}V_0^f = 0, \quad (V_0^s + V_0^f)(0, \cdot) = {}^t\Phi v_0^0(\cdot). \quad (\text{III.98})$$

By construction, the two first lines of (III.97) and (III.98) amount to the same thing as $\mu \partial_{\theta\theta} v_{1,0}^{s\perp} \equiv 0$. We recover here (III.81). From (III.90), we can deduce that $\tilde{\mathcal{A}}V_0^s \equiv \mathcal{A}V_0^s$. Therefore, the discussion concerning (III.98) is the same as the one related to (III.97).

Now, it suffices to observe that the two problems (III.97) and (III.58) are similar. Therefore, the existence and uniqueness of a solution to (III.97) can be obtained by repeating the step $K = 0$ in the proof of Proposition III.5.

◦ *Assume that the condition $\mathcal{HN}(K)$ is true for some $K \in \{0, \dots, N\}$.* Since the criterion (III.81) is satisfied, the problem can be interpreted as before. The matter is to find two functions $V_{K+1}^s := {}^t\Phi v_k^s \in (\mathcal{H}_T^\infty)^{tot}$ and $V_{K+1}^f := {}^t\Phi v_k^f \in (\mathcal{E}_\delta^\infty)^{tot}$ such that:

- For $M \geq 3$:

$$\mathcal{A}V_{K+1}^s = {}^t\Phi f_k^{nl}, \quad \mathcal{B}V_{K+1}^f = {}^t\Phi g_k^{nl}, \quad (V_{K+1}^s + V_{K+1}^f)(0, \cdot) = {}^t\Phi v_{K+1}^0(\cdot).$$

- For $M = 2$:

$$\tilde{\mathcal{A}}V_{K+1}^s = {}^t\Phi f_k^{nl}, \quad \mathcal{B}V_{K+1}^f = {}^t\Phi g_k^{nl}, \quad (V_{K+1}^s + V_{K+1}^f)(0, \cdot) = {}^t\Phi v_{K+1}^0(\cdot).$$

The induction hypothesis applied with the indice K together with Lemmas III.13, III.14 and III.16 say that the functions ${}^t\Phi f_k^{nl}$ and ${}^t\Phi g_k^{nl}$ are known source terms with the expected $(\mathcal{H}_T^\infty)^{tot}$ and $(\mathcal{E}_\delta^\infty)^{tot}$ regularities. When $M \geq 3$, we recognize (III.61) and we can argue as in the proof of Proposition III.5. When $M = 2$, the same types of argument prevail except that, whenever it is necessary, Lemma III.8 must be replaced by Lemma III.15. ■

From the preceding construction, we can also deduce the following information.

Corollary III.17 (Nonlinear homogenization). *For all $k \in \{0, \dots, N+1\}$, the expression Πv_k^s can be determined through the following parabolic equation,*

$$\partial_t \Pi v_k^s - \left(\mu + \frac{1}{\mu} \Pi ((\partial_\theta^{-1} h)^2) \right) \partial_{yy} \Pi v_k^s = S_k^{nl}, \quad (\text{III.99})$$

where the source term $S_k^{nl} := {}^t(S_{1,k}^{nl}, S_{2,k}^{nl})$ depends on the index j with $j < k$. This fact may be formulated by writing ${}^t(S_{1,k}^{nl}, S_{2,k}^{nl}) = {}^t(f_{1,k}^{nl}, f_{2,k}^{nl})(v_0^s, \dots, v_{k-1}^s)$.

2.1.3 Approximated solutions

In this subsection, we prove estimate (III.8). To this end, we explicitly compute the action of the operator \mathcal{L}^a on the approximated solution v_ε^a built in Subsection 2.1.2 and estimate the remainder $R_\varepsilon := \mathcal{L}^a(\varepsilon, v_\varepsilon^a)$ in H^m -norm (for $m \geq 2$). Since $m \geq 2$, $H^m(\mathbb{R}^2)$ is an algebra (we can easily estimate the product of functions), the first step is to decompose the action of \mathcal{L}^a into

$$\mathcal{L}^a(\varepsilon, v_\varepsilon^a(t, \cdot)) = \mathcal{L}^{as}(\varepsilon, v_\varepsilon^{as}(t, \cdot)) + \mathcal{L}^{af}(\varepsilon, v_\varepsilon^{as}(t, \cdot), v_\varepsilon^{af}(t/\varepsilon^2, \cdot)), \quad (\text{III.100})$$

where the operators \mathcal{L}^{as} and \mathcal{L}^{af} are defined in (III.66) and (III.70).

On the one hand, v_ε^{as} is constructed according to the cascade of equations (III.76). We justify further that it is a good approximation for the operator \mathcal{L}^{as} .

On the other hand, v_ε^{af} is constructed so that it approximates the operator \mathcal{L}^{aft} defined at the level of (III.72) instead of \mathcal{L}^{af} . One aspect of the following discussion is to justify that v_ε^{af} is again an approximated solution for the operator \mathcal{L}^{af} . We recall that to pass from operator \mathcal{L}^{af} to \mathcal{L}^{aft} a Taylor formula is applied (to decouple the various time scales). Hence, they only differ from a remainder (which is an integral given by the Taylor formula) that we have to control. To that purpose, we state the following lemma.

Lemma III.18. *Let $m \geq 2$ be an integer and $\delta \in]0, \mu[$. Let $f \in \mathcal{E}_\delta^m(\mathbb{T} \times \mathbb{R})$, $g \in \mathcal{H}_T^{m,0}(\mathbb{T} \times \mathbb{R})$. On the strip $[0, T]$, consider the function $h_{exp}^\varepsilon(t, \cdot) := f(\varepsilon^{-2}t, \cdot) \int_0^t u^N g(u, \cdot) du$. Then the family $\{\varepsilon^{-(N+1)} h_{exp}^\varepsilon\}_\varepsilon$ is bounded in $\mathcal{H}_T^{m,0}(\mathbb{T} \times \mathbb{R})$, i.e.:*

$$\sup_{\varepsilon \in]0,1]} \sup_{t \in [0,T]} \left\| \varepsilon^{-(N+1)} h_{exp}^\varepsilon(t, \cdot) \right\|_{H^m(\mathbb{T} \times \mathbb{R})} < +\infty.$$

Proof of Lemma III.18. We study the competition between the fast decreasing behaviour of the function f and the growth in time of the function $\tilde{g}(t, \cdot) := \int_0^t u^N g(u, \cdot) du$. The product is estimated in homogeneous Sobolev spaces \mathring{H}^j ($j \in \{0, \dots, m\}$) thanks to a Gagliardo-Nirenberg estimate (see Proposition III.19). It requires some L^∞ control on f and \tilde{g} . Let j be an integer such that $j \leq m$.

◦ Since $f \in \mathcal{E}_\delta^m(\mathbb{T} \times \mathbb{R})$, we get:

$$\forall t \in [0, +\infty[, \quad \left\| f(\varepsilon^{-2}t, \cdot) \right\|_{\mathring{H}^m(\mathbb{T} \times \mathbb{R})} \lesssim e^{-\delta \frac{t}{\varepsilon^2}}.$$

From the embedding $H^m(\mathbb{T} \times \mathbb{R}) \hookrightarrow L^\infty(\mathbb{T} \times \mathbb{R})$ ($m \geq 2$), we also obtain

$$\left\| f(\varepsilon^{-2}t, \cdot) \right\|_{L^\infty(\mathbb{T} \times \mathbb{R})} \lesssim e^{-\delta \frac{t}{\varepsilon^2}}, \quad \forall t \in [0, +\infty[.$$

◦ From the assumption on g : $\forall u \in [0, T]$, $\|g(u, \cdot)\|_{L^\infty(\mathbb{T} \times \mathbb{R})} \lesssim \|g(u, \cdot)\|_{H^m(\mathbb{T} \times \mathbb{R})} \leq K$. We use the polynomial structure of the integrand to get a control in L^∞ -norm:

$$\|\tilde{g}(t, \cdot)\|_{L^\infty(\mathbb{T} \times \mathbb{R})} \leq \int_0^t u^N \|g(u, \cdot)\|_{L^\infty(\mathbb{T} \times \mathbb{R})} du \leq K \int_0^t u^N du \leq C_{1,N} t^{N+1},$$

with $C_{1,N} := K/(N+1)$. The control over $\partial^\alpha \tilde{g}$ in L^2 -norm (with $\alpha \in \mathbb{N}^2$ satisfying $|\alpha| = j$) is obtained through

$$\begin{aligned} \|\partial^\alpha \tilde{g}(t, \cdot)\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 &\leq \int_{\mathbb{T} \times \mathbb{R}} \left(\int_0^t u^N \partial^\alpha g(u, \theta, y) du \right)^2 d\theta dy \\ &\leq t \int_0^t u^{2N} \int_{\mathbb{T} \times \mathbb{R}} (\partial^\alpha g(u, \theta, y))^2 d\theta dy du \\ &\leq t \int_0^t u^{2N} \|g(u, \cdot)\|_{H^j(\mathbb{T} \times \mathbb{R})}^2 du \leq tK^2/(2N+1)t^{2N+1} \leq C_{2,N}^2 t^{2N+2}. \end{aligned}$$

with $C_{2,N}^2 := K^2/(2N+1)$.

◦ Finally applying Gagliardo-Nirenberg's inequality leads to for all time $t \in [0, T]$,

$$\begin{aligned} \|f(\varepsilon^{-2}t, \cdot) \tilde{g}(t, \cdot)\|_{\dot{H}^j(\mathbb{T} \times \mathbb{R})} &\lesssim C_g \left(\|f(\varepsilon^{-2}t, \cdot)\|_{\dot{H}^j(\mathbb{T} \times \mathbb{R})} \|\tilde{g}(t, \cdot)\|_{L^\infty(\mathbb{T} \times \mathbb{R})} \right. \\ &\quad \left. + \|f(\varepsilon^{-2}t, \cdot)\|_{L^\infty(\mathbb{T} \times \mathbb{R})} \|\tilde{g}(t, \cdot)\|_{\dot{H}^j(\mathbb{T} \times \mathbb{R})} \right), \\ &\lesssim t^{N+1} e^{-a \frac{t}{\varepsilon^2}}. \end{aligned}$$

The function $\psi_\varepsilon : t \mapsto t^{N+1} e^{-a t/\varepsilon^2}$ is continuous on $[0, T]$ and can be uniformly bounded by:

$$\forall t \in [0, \varepsilon], \quad |\psi_\varepsilon(t)| \leq \varepsilon^{N+1} \quad \text{and} \quad \forall t \in [\varepsilon, T], \quad |\psi_\varepsilon(t)| \leq T^{N+1} e^{-a/\varepsilon}.$$

From the fast decreasing behaviour (to 0) of $e^{-1/\varepsilon}$, we finally deduce that the family $\{\psi_\varepsilon\}_\varepsilon$ is bounded on the strip $[0, T]: \forall t \in [0, T] \quad |\psi_\varepsilon(t)| \lesssim \varepsilon^{N+1}$. ■

•**Proof of property III.8**• We estimate R_ε .

◦ We start by evaluating the first component of R_ε : $\mathcal{L}^{as}(\varepsilon, v_\varepsilon^{as})$. We write v_ε^{as} as the sum $v_\varepsilon^{as} = \sum_{k=0}^{N+1} \varepsilon^k v_k^s$ and we obtain:

$$\begin{aligned} \mathcal{L}^{as}(\varepsilon, v_\varepsilon^{as}) &= \sum_{k=-2}^{N-1} \left(\partial_t v_k^s + \partial_y v_{k+1}^s + \sum_{(i,j) \in \mathcal{J}(M,k+2)} v_{1,i}^s \partial_\theta v_j^s + \sum_{(i,j) \in \mathcal{J}(M,k+1)} v_{2,i}^s \partial_y v_j^s \right. \\ &\quad \left. + {}^t(0, \partial_\theta h v_{1,k+2}^s) - \left(\mu \partial_{\theta\theta} v_{1,k+2}^s + \mu \partial_{yy} v_{1,k}^s + \lambda \partial_{\theta\theta} v_{1,k+1}^s + \lambda \partial_{\theta y} v_{2,k}^s \right) \right) + R_\varepsilon^s, \end{aligned} \tag{III.101}$$

where the remainder R_ε^s is defined as

$$\begin{aligned}
R_\varepsilon^s := & \sum_{k=N}^{N+1} \varepsilon^k \partial_t v_k^s + \varepsilon^N \partial_y v_{N+1}^s \\
& + \sum_{k=N}^{2(N+1)+(M-2)} \varepsilon^k \sum_{(i,j) \in \mathcal{J}(M,k+2)} v_{1,i}^s \partial_\theta v_j^s + \sum_{k=N}^{2(N+1)+(M-1)} \varepsilon^k \sum_{(i,j) \in \mathcal{J}(M,k+1)} v_{2,i}^s \partial_y v_j^s \\
& - \left(\begin{aligned} & \mu \sum_{k=N}^{N+1} \varepsilon^k \partial_{\theta\theta} v_{1,k}^s + \lambda \varepsilon^N \partial_{\theta\theta} v_{1,N+1}^s + \lambda \sum_{k=N}^{N+1} \varepsilon^k \partial_{\theta y} v_{2,k}^s \\ & \mu \sum_{k=N}^{N+1} \varepsilon^k \partial_{\theta\theta} v_{2,k}^s + \lambda \sum_{k=N}^{N+1} \varepsilon^k \partial_{\theta y} v_{1,k}^s + \lambda \sum_{k=N}^{N+2} \varepsilon^k \partial_{yy} v_{2,k-1}^s \end{aligned} \right). \tag{III.102}
\end{aligned}$$

By construction, the family $\{v_k^s\}_{k \in \{0, \dots, N+1\}}$ is solution of the cascade of equations (III.76). The consequence is that the first term of \mathcal{L}^{as} in (III.101) vanishes and $\mathcal{L}^{as}(\varepsilon, v_\varepsilon^s) = R_\varepsilon^s$. Moreover for all $k \in \{0, \dots, N+1\}$, $v_k^s \in \mathcal{H}_T^\infty$ and we can factorize by ε^N in the expression R_ε^s . We get for all interger $m \geq 2$

$$\sup_{\varepsilon \in]0,1]} \varepsilon^{-N} \sup_{t \in [0,T]} \|R_\varepsilon^s(t, \cdot)\|_{H^m(\mathbb{T} \times \mathbb{R})} < +\infty.$$

It justifies that v_ε^s is an approximated solution for the operators \mathcal{L}^{as} (up to order N).

◦ We construct v_ε^f so that it approximates the operator \mathcal{L}^{aft} instead of \mathcal{L}^{af} . Performing a Taylor formula (with respect to the time variable t up to order $N-1$) we have:

$$\mathcal{L}^{af}(\varepsilon, v_\varepsilon^f, v_\varepsilon^s) = \mathcal{L}^{aft}(\varepsilon, v_\varepsilon^f) + R_\varepsilon^{tay}$$

with a remainder which is defined as:

$$\begin{aligned}
R_\varepsilon^{tay}(t, \cdot) := & \varepsilon^{M-2} \left(\int_0^t \frac{u^N}{N!} (\partial_t)^{N-1} v_\varepsilon^{1s}(u, \cdot) du \partial_\theta v_\varepsilon^f(t/\varepsilon^2, \cdot) + \right. \\
& \left. \varepsilon \int_0^t \frac{u^N}{N!} (\partial_t)^{N-1} v_\varepsilon^{2s}(u, \cdot) du \partial_y v_\varepsilon^f(t/\varepsilon^2, \cdot) \right) \\
& + \varepsilon^{M-2} \left(v_\varepsilon^f(t/\varepsilon^2, \cdot) \int_0^t \frac{u^N}{N!} (\partial_t)^{N-1} \partial_\theta v_\varepsilon^s(u, \cdot) du \right. \\
& \left. + v_\varepsilon^f(t/\varepsilon^2, \cdot) \int_0^t \frac{u^N}{N!} (\partial_t)^{N-1} \partial_y v_\varepsilon^s(u, \cdot) du \right).
\end{aligned}$$

These terms are of the form of h_{exp}^ε defined at the level of Lemma III.18. According to Proposition III.1, the hypothesis of Lemma III.18 are satisfied, for all integer $m \geq 2$ we have:

$$\sup_{\varepsilon \in]0,1]} \sup_{t \in [0,T]} \varepsilon^{-N} \|R_\varepsilon^{tay}(t, \cdot)\|_{H^m(\mathbb{T} \times \mathbb{R})} < +\infty.$$

◦ Finally, there remains to understand the last contribution $\mathcal{L}^{aft}(\varepsilon, v_\varepsilon^f)$. Plugging $v_\varepsilon^s = \sum_{k=0}^{N+1} \varepsilon^k v_k^s$

and $v_\varepsilon^f = \sum_{k=0}^{N+1} \varepsilon^k v_k^f$ into \mathcal{L}^{aft} leads to:

$$\begin{aligned}
\mathcal{L}^{aft}(\varepsilon, v_\varepsilon^f)(t/\varepsilon^2) &= \sum_{k=-2}^{N-1} \varepsilon^k \left(\partial_\tau v_{k+2}^f(t/\varepsilon^2, \cdot) + h \partial_y v_{k+1}^f(t/\varepsilon^2, \cdot) + {}^t(0, \partial_\theta h v_{1,k+2}^f(t/\varepsilon^2, \cdot)) \right. \\
&+ \sum_{(i,j) \in \mathcal{J}(M,k+2)} v_{1,i}^s(t/\varepsilon^2, \cdot) \partial_\theta v_j^f(t/\varepsilon^2, \cdot) + \sum_{(i,j) \in \mathcal{J}(M,k+1)} v_{2,i}^f(t/\varepsilon^2, \cdot) \partial_y v_j^f(t/\varepsilon^2, \cdot) \\
&+ \sum_{(i,j,l) \in \mathcal{I}(M,k+2)} \frac{t^l}{\varepsilon^{2l} l!} \partial_t^l v_{1,i}^s(0, \cdot) \partial_\theta v_j^f(t/\varepsilon^2, \cdot) + \sum_{(i,j,l) \in \mathcal{I}(M,k+1)} \frac{t^l}{\varepsilon^{2l} l!} \partial_t^l v_{2,i}^s(0, \cdot) \partial_y v_j^f(t/\varepsilon^2, \cdot) \\
&+ \sum_{(i,j,l) \in \mathcal{I}(M,k+2)} \frac{t^l}{\varepsilon^{2l} l!} v_{1,i}^f(t/\varepsilon^2, \cdot) \partial_t^l \partial_\theta v_j^s(0, \cdot) + \sum_{(i,j,l) \in \mathcal{I}(M,k+1)} \frac{t^l}{\varepsilon^{2l} l!} v_{1,i}^f(t/\varepsilon^2, \cdot) \partial_t^l \partial_y v_j^s(0, \cdot) \Big) \\
&- \left(\mu \sum_{k=N}^{N+1} \varepsilon^k \partial_{\theta\theta} v_{1,f}^f(t/\varepsilon^2, \cdot) + \lambda \varepsilon^N \partial_{\theta\theta} v_{1,N+1}^f(t/\varepsilon^2, \cdot) + \lambda \sum_{k=N}^{N+1} \varepsilon^k \partial_{\theta y} v_{2,k}^f(t/\varepsilon^2, \cdot) \right. \\
&\left. \mu \sum_{k=N}^{N+1} \varepsilon^k \partial_{\theta\theta} v_{2,k}^f(t/\varepsilon^2, \cdot) + \lambda \sum_{k=N}^{N+1} \varepsilon^k \partial_{\theta y} v_{1,k}^f(t/\varepsilon^2, \cdot) + \lambda \sum_{k=N}^{N+2} \varepsilon^k \partial_{yy} v_{2,k-1}^f(t/\varepsilon^2, \cdot) \right) + R_\varepsilon^f,
\end{aligned} \tag{III.103}$$

where R_ε^f the remainder can be decomposed as $R_\varepsilon^f := R_\varepsilon^{1f} + R_\varepsilon^{2f}$ with R_ε^{1f} defined by

$$\begin{aligned}
R_\varepsilon^{1f} &:= \varepsilon^N \partial_y v_{N+1}^f(t/\varepsilon^2, \cdot) + \sum_{k=N}^{2(N+1)+(M-2)} \varepsilon^k \sum_{(i,j) \in \mathcal{J}(M,k+2)} v_{1,i}^f(t/\varepsilon^2, \cdot) \partial_\theta v_j^f(t/\varepsilon^2, \cdot) \\
&+ \sum_{k=N}^{2(N+1)+(M-1)} \varepsilon^k \sum_{(i,j) \in \mathcal{J}(M+1,k+2)} v_{2,i}^f(t/\varepsilon^2, \cdot) \partial_y v_j^f(t/\varepsilon^2, \cdot) \\
&- \left(\mu \sum_{k=N}^{N+1} \varepsilon^k \partial_{\theta\theta} v_{1,k}^f(t/\varepsilon^2, \cdot) + \lambda \varepsilon^N \partial_{\theta\theta} v_{1,N+1}^f(t/\varepsilon^2, \cdot) + \lambda \sum_{k=N}^{N+1} \varepsilon^k \partial_{\theta y} v_{2,k}^f(t/\varepsilon^2, \cdot) \right. \\
&\left. \mu \sum_{k=N}^{N+1} \varepsilon^k \partial_{\theta\theta} v_{2,k}^f(t/\varepsilon^2, \cdot) + \lambda \sum_{k=N}^{N+1} \varepsilon^k \partial_{\theta y} v_{1,k}^f(t/\varepsilon^2, \cdot) + \lambda \sum_{k=N}^{N+2} \varepsilon^k \partial_{yy} v_{2,k-1}^f(t/\varepsilon^2, \cdot) \right),
\end{aligned}$$

whereas R_ε^{2f} consists of the terms corresponding to the Taylor formula:

$$\begin{aligned} R_\varepsilon^{2f} := & \sum_{k=N}^{4N+(M-2)} \varepsilon^k \left(\sum_{(i,j,l) \in \mathcal{I}(M,k+2)} \frac{t^l}{\varepsilon^{2l} l!} \partial_t^l v_{1,i}^s(0, \cdot) \partial_\theta v_j^f(t/\varepsilon^2, \cdot) \right. \\ & \left. + \sum_{(i,j,l) \in \mathcal{I}(M,k+2)} \frac{t^l}{\varepsilon^{2l} l!} v_{1,i}^f(t/\varepsilon^2, \cdot) \partial_t^l \partial_\theta v_j^s(0, \cdot) \right) \\ & + \sum_{k=N}^{4N+(M-1)} \varepsilon^k \left(\sum_{(i,j,l) \in \mathcal{I}(M,k+1)} \frac{t^l}{\varepsilon^{2l} l!} \partial_t^l v_{2,i}^s(0, \cdot) \partial_y v_j^f(t/\varepsilon^2, \cdot) \right. \\ & \left. + \sum_{(i,j,l) \in \mathcal{I}(M,k+1)} \frac{t^l}{\varepsilon^{2l} l!} v_{1,i}^f(t/\varepsilon^2, \cdot) \partial_t^l \partial_y v_j^s(0, \cdot) \right). \end{aligned}$$

First of all, since $\{v_k^f\}_{k \in \llbracket 0, N+1 \rrbracket}$ satisfies the cascade of equation (III.77), some contributions vanish in (III.103):

$$\mathcal{L}^{aft}(\varepsilon, v_\varepsilon^f)(t/\varepsilon^2) = R_\varepsilon^{1f} + R_\varepsilon^{2f}.$$

-We start by estimating R_ε^{1f} . By construction (see Proposition III.1), the profiles $\{v_k^f\}_{k \in \{0, \dots, N+1\}}$ lie in $\mathcal{E}_\delta^\infty$. Furthermore, we can factorize R_ε^{1f} by ε^N . We can write, for all integer $m \geq 2$:

$$\sup_{\varepsilon \in]0,1]} \sup_{t \in [0,T]} \varepsilon^{-N} \left\| R_\varepsilon^{1f}(t, \cdot) \right\|_{H^m(\mathbb{T} \times \mathbb{R})} < +\infty.$$

-There remains the term R_ε^{2f} . *A priori* it can be dangerous because it contains some polynomials in the variable t/ε^2 . Nevertheless we use the fast decreasing behaviours of the profile v_ε^f . Indeed if $f \in \mathcal{E}_\delta^\infty$ then for all $l \in \mathbb{N}$ the function $\tau^l f$ is also fastly decreasing: $\tau^l f \in \mathcal{E}_{\delta'}^\infty$ for some $0 < \delta' < \delta$. Furthermore, noticing that we can factorize by ε^N in R_ε^{2f} , we obtain for all integer $m \geq 2$:

$$\sup_{\varepsilon \in]0,1]} \sup_{t \in [0,T]} \varepsilon^{-N} \left\| R_\varepsilon^{2f}(t, \cdot) \right\|_{H^m(\mathbb{T} \times \mathbb{R})} < +\infty. \quad \blacksquare$$

2.2 The case of the pressure - Consequence

In this subsection we still assume that $\lambda_\varepsilon = \lambda \varepsilon$ and $M \geq 2$. First of all we quickly prove Proposition III.2. Then we take advantage of the control obtained on $\{q_\varepsilon^a\}_\varepsilon$ to prove that the approximated solution v_ε^a is a good approximation for operator $(\mathcal{L}^1, \mathcal{L}^2)$ assuming ν is large enough (Proposition III.3).

2.2.1 Approximated pressure - Proof of Proposition III.2

In Subsection 2.2.1, we prove Proposition III.2. Consider the approximated velocity v_ε^a built according to Proposition III.1. The existence and uniqueness of a pressure q_ε^a solution of (III.10) with a control over the family (III.11) in H^s is classical. The obtention of (III.11)

follows.

Since the operator \mathcal{L}_0 is linear with respect to the pressure variable, we build the profile q_k^ε as the solution of

$$\begin{aligned} \mathcal{L}_0(\varepsilon, q_k^\varepsilon, v_\varepsilon^a) &= \partial_t q_k^\varepsilon + \varepsilon^{-1} h \partial_y q_k^\varepsilon \\ &\quad + \varepsilon^{M-2} (v_\varepsilon^{a1} \partial_\theta q_k^\varepsilon + \varepsilon v_\varepsilon^{2a} \partial_y q_k^\varepsilon) + C \varepsilon^{M-2} q_k^\varepsilon (\partial_\theta v_\varepsilon^{a1} + \varepsilon \partial_y v_\varepsilon^{2a}) = 0 \end{aligned} \quad (\text{III.104})$$

with initial data satisfying $q_k^\varepsilon(0, \cdot) = q_k^0(\cdot)$. We recover the approximated solution q_ε^a by summing over all multi-indices $k \in \llbracket 0, N+1 \rrbracket$: $q_\varepsilon^a := \sum_{k=0}^{N+1} \varepsilon^k q_k^\varepsilon$.

At fixed ε , the system (III.104) can be written under the form:

$$\partial_t \dot{q} + f \cdot \nabla \dot{q} + g \dot{q} = 0,$$

for some f and g lying in \mathcal{H}_T^∞ and for all $T \in \mathbb{R}_+^*$. This is a transport equation with coefficients lying in \mathcal{W}_T^∞ . For all time $T \in \mathbb{R}_+^*$ there exists a unique solution in \mathcal{H}_T^∞ satisfying the initial data $q_k^\varepsilon(0, \cdot) = q_k^0(\cdot)$. Thus (III.10) is satisfied.

There remains to bound the family of functions $\{q_k^\varepsilon\}_\varepsilon$. Consider the simplified equation:

$$\partial_t \check{q}_\varepsilon + \varepsilon^{-1} h \partial_y \check{q}_\varepsilon = 0, \quad \check{q}(0, \cdot) = q_k^0(\cdot). \quad (\text{III.105})$$

The solution is explicit $\check{q}_\varepsilon(t, \theta, y) = q_k^0(\theta, y - \varepsilon^{-1} h(\theta) t)$. The family $\{\check{q}_\varepsilon\}_\varepsilon$ is bounded in L^2 . The family $\{\partial_\theta \check{q}_\varepsilon\}_\varepsilon$ is not bounded whereas the family $\{\varepsilon \partial_\theta \check{q}_\varepsilon\}_\varepsilon$ is. To get rid of this loss of regularity when derivating with respect to ∂_θ we introduce anisotropic Sobolev spaces $H_{(1,\varepsilon)}^m$. Let α be a multi-index of length m . Derivating (III.104) with respect to $\varepsilon^\alpha \partial^\alpha$, multiplying by $\varepsilon^\alpha \partial^\alpha q_k^\varepsilon$ and integrating (that is performing classical hyperbolic estimates in a weighted version of L^2) we obtain for all time $T \in \mathbb{R}_+^*$ that there exists a positive constant C (independent of ε) such that:

$$\forall \varepsilon \in]0, 1], \quad \forall t \in [0, T], \quad \|q_k^\varepsilon(t, \cdot)\|_{H_{(1,\varepsilon)}^m} \lesssim e^{Ct} \|q_k^0\|_{H^m}.$$

That is to say Inequality (III.11) is satisfied. ■

2.2.2 Approximated solutions for operator \mathcal{L}

One consequence of the control obtained on the family $\{q_\varepsilon^a\}_\varepsilon$ is that $\{(q_\varepsilon^a, v_\varepsilon^a)\}_\varepsilon$ is an approximated solution for the operator \mathcal{L} , *i.e.* Proposition III.3 is satisfied.

Consider $(q_\varepsilon^a, v_\varepsilon^a)$ built in Subsections 2.1 and 2.2. By construction, it satisfies $\mathcal{L}_0(\varepsilon, q_\varepsilon^a, v_\varepsilon^a) = 0$ and:

$${}^t(\mathcal{L}_1, \mathcal{L}_2)(\varepsilon, q_\varepsilon^a, v_\varepsilon^a) = \mathcal{L}^a(\varepsilon, v_\varepsilon^a) + C \varepsilon^{2\nu-M-2} {}^t(q_\varepsilon^a \partial_\theta q_\varepsilon^a, \varepsilon q_\varepsilon^a \partial_y q_\varepsilon^a).$$

The quantity $\mathcal{L}^a(\varepsilon, v_\varepsilon^a)$ can be estimated thanks to (III.8). Then we estimate

$$\varepsilon^{2\nu-M-2} {}^t(q_\varepsilon^a \partial_\theta q_\varepsilon^a, \varepsilon q_\varepsilon^a \partial_y q_\varepsilon^a)$$

thanks to the Gagliardo-Nirenberg's estimate. Let $\alpha \in \mathbb{N}^2$, $|\alpha| \leq m$, then

$$\|\partial^\alpha (q_\varepsilon^a \partial_\theta q_\varepsilon^a)\|_{L^2} = \left\| \partial^\alpha \partial_\theta (q_\varepsilon^a)^2 \right\|_{L^2} \leq 2C_g \|q_\varepsilon^a\|_{L^\infty} \|q_\varepsilon^a\|_{\dot{H}^{|\alpha|+1}}.$$

Yet, we only get a control of the pressure in the anisotropic version of the Sobolev spaces. First we use (III.5) to bound the L^∞ norm. Then, we apply the following equivalence of norms:

$$\|\cdot\|_{H^m} \leq \varepsilon^{-m} \|\cdot\|_{H_{(1,\varepsilon)}^m}. \quad (\text{III.106})$$

Thus we obtain that:

$$\|\partial^\alpha (q_\varepsilon^a \partial_\theta q_\varepsilon^a)\|_{L^2} \leq 2C_g \varepsilon^{-1/2} \varepsilon^{m+1} \|q_\varepsilon^a\|_{H_{(1,\varepsilon)}^{m+1}}^2.$$

Finally the non-linear term involving the pressure can be estimated as follows:

$$\varepsilon^{-N} \left\| \varepsilon^{2\nu-M-2} t (q_\varepsilon^a \partial_\theta q_\varepsilon^a, \varepsilon q_\varepsilon^a \partial_y q_\varepsilon^a) \right\|_{H^m} \lesssim \varepsilon^{2\nu-M-5/2-(m+1)-N} \|q_\varepsilon^a\|_{H_{(1,\varepsilon)}^{m+1}}^2 \lesssim \varepsilon^{2\nu-M-5/2-(m+1)-N}.$$

Assuming (III.12), it completes the proof. ■

3 Energy estimates

In this section we prove the result of existence and stability (Theorem III.4). We fix m, ν and M three integers satisfying:

$$M \geq 7/2 \quad \text{and} \quad 2\nu - M - 5/2 - (m+3) \geq 0. \quad (\text{III.107})$$

Here, we still assume that $\lambda_\varepsilon := \varepsilon\lambda$. All the estimates are still true for general λ_ε providing that we can build an approximated solution.

Let $(q_\varepsilon^a, v_\varepsilon^a)$ an approximated solution of order N constructed on the interval $[0, 1]$ according to Proposition III.1 and Proposition III.2. The family $\{(q_\varepsilon^a, v_\varepsilon^a)\}_\varepsilon$ lies in $\mathcal{H}_{1,(1,\varepsilon)}^{m+6,0} \times \mathcal{H}_1^{m+6,0}$ and satisfies Inequalities (III.8) and (III.11). We denote by C_a , $C_{\mathcal{L}^a}$ and $C_{\mathcal{L}}$ positive constants such that:

$$\sup_{\varepsilon \in]0,1]} \sup_{t \in [0,1]} \|v_\varepsilon^a(t, \cdot)\|_{H^{m+6}} < C_a, \quad \sup_{\varepsilon \in]0,1]} \sup_{t \in [0,1]} \|q_\varepsilon^a(t, \cdot)\|_{H_{(1,\varepsilon)}^{m+6}} < C_a, \quad (\text{III.108})$$

and

$$\sup_{\varepsilon \in]0,1]} \sup_{t \in [0,1]} \|\varepsilon^{-N} \mathcal{L}^a(\varepsilon, v_\varepsilon^a)\|_{H^{m+6}(\mathbb{T} \times \mathbb{R})} < C_{\mathcal{L}^a}, \quad (\text{III.109})$$

$$\sup_{\varepsilon \in]0,1]} \sup_{t \in [0,1]} \|\varepsilon^{-N} \mathcal{L}_0(\varepsilon, q_\varepsilon^a, v_\varepsilon^a)\|_{H_{(1,\varepsilon)}^{m+6}(\mathbb{T} \times \mathbb{R})} < C_{\mathcal{L}}. \quad (\text{III.110})$$

Since \mathcal{L} is symmetrisable, we recall that for any $\varepsilon \in]0, 1]$ there exists $T_\varepsilon \in \mathbb{R}_+^*$ such that $(q_\varepsilon^e, v_\varepsilon^e)$ is solution in H^m ($m > 2$) of (III.6) with initial data satisfying (III.7) and (III.10) (see for instance [39, 40]). Let $R \in \mathbb{N}$, we define on the strip $[0, \min(1, T_\varepsilon)]$ the functions

q_ε^R and v_ε^R through the identity $(q_\varepsilon^e, v_\varepsilon^e) := (q_\varepsilon^a, v_\varepsilon^a) + \varepsilon^R(q_\varepsilon^R, v_\varepsilon^R)$. Instead of working on the system (III.1)-(III.2) to prove the existence of a time T_c (independent of ε) such that:

$$\forall \varepsilon \in]0, 1], \quad \min(1, T_\varepsilon) \geq T_c$$

we prefer studying the new variable $(q_\varepsilon^R, v_\varepsilon^R)$. We justify an accurate control over the growth in time of the solution for the norm $H_{(1,\varepsilon)}^m \times H^m$ (see Proposition III.20). To this end we perform an energy method on the singular system (with respect to ε) satisfied by $(q_\varepsilon^r, v_\varepsilon^r)$:

$$\begin{aligned} \partial_t q_\varepsilon^R + \varepsilon^{-1} h \partial_y q_\varepsilon^R + \varepsilon^{M-2} (v_\varepsilon^{1a} \partial_\theta q_\varepsilon^R + \varepsilon v_\varepsilon^{2a} \partial_y q_\varepsilon^R) + C \varepsilon^{M-2} q_\varepsilon^R (\partial_\theta v_\varepsilon^{1a} + \varepsilon \partial_y v_\varepsilon^{2a}) \\ + \varepsilon^{M-2} (v_\varepsilon^{1R} \partial_\theta q_\varepsilon^a + \varepsilon v_\varepsilon^{2R} \partial_y q_\varepsilon^a) + C \varepsilon^{M-2} q_\varepsilon^a (\partial_\theta v_\varepsilon^{1R} + \varepsilon \partial_y v_\varepsilon^{2R}) = S_\varepsilon^{0,R,N}, \end{aligned} \quad (\text{III.111a})$$

$$\begin{aligned} \partial_t v_\varepsilon^R + \varepsilon^{-1} h \partial_y v_\varepsilon^R + \varepsilon^{M-2} (v_\varepsilon^{1a} \partial_\theta v_\varepsilon^R + \varepsilon v_\varepsilon^{2a} \partial_y v_\varepsilon^R) + \varepsilon^{M-2} (v_\varepsilon^{1R} \partial_\theta v_\varepsilon^a + \varepsilon v_\varepsilon^{2R} \partial_y v_\varepsilon^a) \\ + \boxed{t(0, \varepsilon^{-2} \partial_\theta h v_\varepsilon^{1R})} - \tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon}(v_\varepsilon^R) = S_\varepsilon^{R,N}, \end{aligned} \quad (\text{III.111b})$$

with initial data $(q_\varepsilon^R(0, \cdot), v_\varepsilon^R(0, \cdot)) \equiv 0$. The term $(S_\varepsilon^{0,R,N}, S_\varepsilon^{R,N})$ can be computed as a nonlinear function of $(q_\varepsilon^R, v_\varepsilon^R)$:

$$\begin{aligned} S_\varepsilon^{0,R,N} &:= -\varepsilon^{N-R}(\varepsilon^{-N} \mathcal{L}_0(\varepsilon, q_\varepsilon^a, v_\varepsilon^a)) - \varepsilon^{R+M-2} (v_\varepsilon^{1R} \partial_\theta q_\varepsilon^R + \varepsilon v_\varepsilon^{2R} \partial_y q_\varepsilon^R) \\ &\quad - C \varepsilon^{R+M-2} q_\varepsilon^R (\partial_\theta v_\varepsilon^{1R} + \varepsilon \partial_y v_\varepsilon^{2R}), \\ S_\varepsilon^{R,N} &:= -\varepsilon^{N-R}(\varepsilon^{-N} \mathcal{L}^a(\varepsilon, v_\varepsilon^a)) - \varepsilon^{R+M-2} (v_\varepsilon^{1R} \partial_\theta v_\varepsilon^R + \varepsilon v_\varepsilon^{2R} \partial_y v_\varepsilon^R) \\ &\quad - \frac{C \varepsilon^{2\nu-M-R-2}}{2} t(\partial_\theta, \varepsilon \partial_y) (q_\varepsilon^a + \varepsilon^R q_\varepsilon^R)^2. \end{aligned}$$

In addition, the control obtained on $(q_\varepsilon^R, v_\varepsilon^R)$ justifies that $(q_\varepsilon^a, v_\varepsilon^a)$ is a good approximation of $(q_\varepsilon^e, v_\varepsilon^e)$ (in well chosen norms) up to order R .

•**A pure hyperbolic approach**• We first assume $\mu = \lambda = 0$, so the dissipation $\tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon}$ vanishes. Let $(q_\varepsilon^l, v_\varepsilon^l)$ be solution of the linearized equation $\mathcal{L}(\varepsilon, q_\varepsilon, v_\varepsilon) = 0$ in $(q_\varepsilon^a, v_\varepsilon^a)$:

$$\begin{aligned} \partial_t q_\varepsilon^l + \varepsilon^{-1} h \partial_y q_\varepsilon^l + \varepsilon^{M-2} (v_\varepsilon^a \partial_\theta q_\varepsilon^l + \varepsilon v_\varepsilon^a \partial_y q_\varepsilon^l) + C \varepsilon^{M-2} q_\varepsilon^l (\partial_\theta v_\varepsilon^a + \varepsilon \partial_y v_\varepsilon^a) \\ + \varepsilon^{M-2} (v_\varepsilon^{1l} \partial_\theta q_\varepsilon^a + \varepsilon v_\varepsilon^{2l} \partial_y q_\varepsilon^a) + C \varepsilon^{M-2} q_\varepsilon^a (\partial_\theta v_\varepsilon^{1l} + \varepsilon \partial_y v_\varepsilon^{2l}) = -\mathcal{L}_0(\varepsilon, q_\varepsilon^a, v_\varepsilon^a), \end{aligned} \quad (\text{III.112a})$$

$$\begin{aligned} \partial_t v_\varepsilon^l + \varepsilon^{-1} h \partial_y v_\varepsilon^l + \varepsilon^{M-2} (v_\varepsilon^{1a} \partial_\theta v_\varepsilon^l + \varepsilon v_\varepsilon^{2a} \partial_y v_\varepsilon^l) + \varepsilon^{M-2} (v_\varepsilon^{1l} \partial_\theta v_\varepsilon^a + \varepsilon v_\varepsilon^{2l} \partial_y v_\varepsilon^a) \\ + \boxed{t(0, \varepsilon^{-2} \partial_\theta h v_\varepsilon^{1l})} = -C \varepsilon^{2\nu-2} (q_\varepsilon^a t(\partial_\theta, \varepsilon \partial_y) q_\varepsilon^l + q_\varepsilon^l t(\partial_\theta, \varepsilon \partial_y) q_\varepsilon^a) - t(\mathcal{L}^1, \mathcal{L}^2)(\varepsilon, v_\varepsilon^a), \end{aligned} \quad (\text{III.112b})$$

with initial data $(q_\varepsilon^l(0, \cdot), v_\varepsilon^l(0, \cdot)) \equiv 0$. At fixed ε the system has a unique solution on $[0, T_\varepsilon^l]$. We perform a classical energy method (for symmetrizable and hyperbolic systems) on (III.112★). First, we multiply (III.112a)-(III.112b) by $t(\varepsilon^{2\nu-M} q_\varepsilon^l, v_\varepsilon^l)$ to eliminate terms such as

$$C \varepsilon^{M-2} q_\varepsilon^a (\partial_\theta v_\varepsilon^{1l} + \varepsilon \partial_y v_\varepsilon^{2l}) \quad \text{and} \quad C \varepsilon^{2\nu-2} q_\varepsilon^a t(\partial_\theta, \varepsilon \partial_y) q_\varepsilon^l.$$

Then we integrate with respect to θ and y and apply Gronwall's lemma to obtain

$$\begin{aligned} \forall t \in [0, T_\varepsilon^l], \quad & \left\| \left(\varepsilon^{2\nu-M} q_\varepsilon^l, v_\varepsilon^l \right) (t, \cdot) \right\|_{L^2(\mathbb{T} \times \mathbb{R})} \\ & \lesssim e^{C_\varepsilon t} \sup_{t \in [0, 1]} \left\| \left(\varepsilon^{2\nu-M} \mathcal{L}_0(\varepsilon, q_\varepsilon^a, v_\varepsilon^a), \mathcal{L}^a(\varepsilon, v_\varepsilon^a) \right) \right\|_{L^2(\mathbb{T} \times \mathbb{R})}, \end{aligned} \quad (\text{III.113})$$

with

$$C_\varepsilon \leq C (1 + \varepsilon^{-2} \|\partial_\theta h\|_{L^\infty}). \quad (\text{III.114})$$

Inequality (III.114) indicates that the weighted (in ε) L^2 -norm of the linearized solution can increase exponentially with t . It seems to remain bounded for time T_ε^l of order ε^2 . This issue has already been pointed out in several works dealing with turbulence [11, 12, 21]. They underline the difficulty to describe the solution in a classical way in long time ($t \approx 1$). In [12], to deal with the singular term $\varepsilon^{-2} \|\partial_\theta h\|_{L^\infty}$ (in Inequality (III.114)), a change of unknown is considered to absorb the boxed term $\varepsilon^{-2} \partial_\theta h v_\varepsilon^{1l}$ in Equation (III.112b). However, such a change is not compatible with the definition of the viscosity term $\tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon}$.

Here, with no more information, a *pure hyperbolic method* only provides a control of the solution up to a time of order ε^2 .

• **Role of the dissipation** • To go further, we need to take into account the parabolic behaviours of Equation (III.111b). We add the dissipation $-\tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon}$ in the linear system (III.112★):

$$\begin{aligned} & \partial_t q_\varepsilon^l + \varepsilon^{-1} h \partial_y q_\varepsilon^l + \varepsilon^{M-2} \left(v_\varepsilon^a \partial_\theta q_\varepsilon^l + \varepsilon v_\varepsilon^a \partial_y q_\varepsilon^l \right) + C \varepsilon^{M-2} q_\varepsilon^l (\partial_\theta v_\varepsilon^a + \varepsilon \partial_y v_\varepsilon^a) \\ & - \varepsilon^{M-2} \left(v_\varepsilon^{1l} \partial_\theta q_\varepsilon^a + \varepsilon v_\varepsilon^{2l} \partial_y q_\varepsilon^a \right) - C \varepsilon^{M-2} q_\varepsilon^a \left(\partial_\theta v_\varepsilon^{1l} + \varepsilon \partial_y v_\varepsilon^{2l} \right) = -\mathcal{L}_0(\varepsilon, q_\varepsilon^a, v_\varepsilon^a), \quad (\text{III.115a}) \\ & \partial_t v_\varepsilon^l + \varepsilon^{-1} h \partial_y v_\varepsilon^l + \varepsilon^{M-2} \left(v_\varepsilon^{1a} \partial_\theta v_\varepsilon^l + \varepsilon v_\varepsilon^{2a} \partial_y v_\varepsilon^l \right) + \varepsilon^{M-2} \left(v_\varepsilon^{1l} \partial_\theta v_\varepsilon^a + \varepsilon v_\varepsilon^{2l} \partial_y v_\varepsilon^a \right) \\ & + \boxed{t(0, \varepsilon^{-2} \partial_\theta h v_\varepsilon^{1l})} - \tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon} v_\varepsilon^l = -\frac{C \varepsilon^{2\nu-2}}{2} \left(q_\varepsilon^a \partial_\theta v_\varepsilon^l + q_\varepsilon^l \partial_\theta v_\varepsilon^a \right) - \mathcal{L}^a(\varepsilon, v_\varepsilon^a). \end{aligned} \quad (\text{III.115b})$$

The viscosity $-\tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon}$ is a non-negative operator. Furthermore there exists c a positive constant such that for any function $f \in H^1(\mathbb{T} \times \mathbb{R})$,

$$\forall \varepsilon \in]0, 1], \quad -\left\langle \tilde{\mathcal{P}}_{\varepsilon, \lambda_\varepsilon} f, f \right\rangle \geq c \left(\left\| \varepsilon^{-1} \partial_\theta f \right\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 + \left\| \partial_y f \right\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 \right) := \Phi_\varepsilon(\nabla, f). \quad (\text{III.116})$$

Taking this new estimate into account in the energy inequality has two consequences.

- **Absorption phenomena** • First at fixed ε , it allows to absorb terms such as $\partial_\theta v_\varepsilon^R$ and $\partial_y v_\varepsilon^R$ in L^2 -norm. Thereby, we do not have to multiply the pressure by $\varepsilon^{2\nu-M}$ to eliminate terms such as

$$C \varepsilon^{M-2} q_\varepsilon^a \left(\partial_\theta v_\varepsilon^{1l} + \varepsilon \partial_y v_\varepsilon^{2l} \right) \quad \text{and} \quad C \varepsilon^{2\nu-2} q_\varepsilon^a \partial_\theta v_\varepsilon^l$$

by an argument of symmetry. Thus we obtain L^2 estimates for the velocity instead of weighted (in ε) L^2 estimates.

Furthermore, taking the dependence in ε into account, the presence of the singular term $\varepsilon^{-1} \partial_\theta v_\varepsilon^R$ indicates that we can expect some absorption of the singular term $\varepsilon^{-2} \partial_\theta h v_\varepsilon^{1R}$.

- **Regularization phenomena** • A second aspect is that in addition to having estimates $L_t^\infty L_{\theta,y}^2(\mathbb{T} \times \mathbb{R})$ we expect some regularization of the velocity. It should belong to $L_t^2 H_{\theta,y}^1(\mathbb{T} \times \mathbb{R})$.

Having the absorption phenomena in mind we can hope for a more accurate estimate for C_ε . Indeed the boxed term $\varepsilon^{-2} \partial_\theta h v_\varepsilon^{1l}$ can be estimated as

$$\varepsilon^{-1} \left| \varepsilon^{-1} \int_{\mathbb{T} \times \mathbb{R}} \partial_\theta h v_\varepsilon^{1l} v_\varepsilon^{2l} d\theta dy \right| \lesssim \varepsilon^{-1} \left(\|h\|_{L^\infty}^2 \|v_\varepsilon^l\|_{L^2}^2 + \|\varepsilon^{-1} \partial_\theta v_\varepsilon^l\|_{L^2}^2 \right).$$

This estimates indicates that the term $\varepsilon^{-2} \partial_\theta h v_\varepsilon^l$ is thereby singular of order ε^{-1} rather than being singular of order ε^{-2} . Then, we can hope for a control for time of order ε .

- **A singular change of unknowns** • Finally to get rid of this singular term (of order 1) we consider a family (in ε) of singular changes of variables. We naively consider the changes of variables:

$$\tilde{q}_\varepsilon^R := q_\varepsilon^R, \quad \tilde{v}_\varepsilon^{1R} := v_\varepsilon^{1R}, \quad \tilde{v}_\varepsilon^{2R} := \varepsilon v_\varepsilon^{2R}.$$

In some way, it strengthens the influence of v_ε^{2R} in the system (or diminishes the influence of v_ε^{1R}). Injecting $\tilde{v}_\varepsilon^{2R} := \varepsilon v_\varepsilon^{2R}$ in the second Equation (III.115b) brings a negative power of ε in front of the derivative in time ∂_t . We eliminate it by multiplying by ε . System (III.111) can be written:

$$\begin{aligned} \partial_t \tilde{q}_\varepsilon^R + \varepsilon^{-1} h \partial_y \tilde{q}_\varepsilon^R + \varepsilon^{M-2} (v_\varepsilon^{1a} \partial_\theta \tilde{q}_\varepsilon^R + \varepsilon v_\varepsilon^{2a} \partial_y \tilde{q}_\varepsilon^R) + C \varepsilon^{M-2} \tilde{q}_\varepsilon^R (\partial_\theta v_\varepsilon^{1a} + \varepsilon \partial_y v_\varepsilon^{2a}) \\ + \varepsilon^{M-2} (\tilde{v}_\varepsilon^{1R} \partial_\theta q_\varepsilon^a + \tilde{v}_\varepsilon^{2R} \partial_y q_\varepsilon^a) + C \varepsilon^{M-2} q_\varepsilon^a (\partial_\theta \tilde{v}_\varepsilon^{1R} + \partial_y \tilde{v}_\varepsilon^{2R}) = \tilde{S}_\varepsilon^{0,R,N}, \end{aligned} \quad (\text{III.117a})$$

$$\begin{aligned} \partial_t \tilde{v}_\varepsilon^R + \varepsilon^{-1} h \partial_y \tilde{v}_\varepsilon^R + \boxed{t (0, \varepsilon^{-1} \partial_\theta h \tilde{v}_\varepsilon^{1R})} + \varepsilon^{M-2} (v_\varepsilon^{1a} \partial_\theta \tilde{v}_\varepsilon^R + \varepsilon v_\varepsilon^{2a} \partial_y \tilde{v}_\varepsilon^R) \\ + \varepsilon^{M-2} (\tilde{v}_\varepsilon^{1R} \partial_\theta + \tilde{v}_\varepsilon^{2R} \partial_y) t (v_\varepsilon^{1a}, \varepsilon v_\varepsilon^{2a}) - \mathcal{Q}_{\varepsilon, \lambda_\varepsilon}(\tilde{v}_\varepsilon^R) = \tilde{S}_\varepsilon^{R,N}, \end{aligned} \quad (\text{III.117b})$$

together with the initial data $(q_\varepsilon^R(0, \cdot), v_\varepsilon^R(0, \cdot)) \equiv 0$. The new sources are:

$$\begin{aligned} \tilde{S}_\varepsilon^{0,R,N} &:= -\varepsilon^{N-R} (\varepsilon^{-N} \mathcal{L}_0(\varepsilon, q_\varepsilon^a, v_\varepsilon^a)) - \varepsilon^{R+M-2} (\tilde{v}_\varepsilon^{1R} \partial_\theta \tilde{q}_\varepsilon^R + \tilde{v}_\varepsilon^{2R} \partial_y \tilde{q}_\varepsilon^R) \\ &\quad - C \varepsilon^{R+M-2} \tilde{q}_\varepsilon^R (\partial_\theta \tilde{v}_\varepsilon^{1R} + \partial_y \tilde{v}_\varepsilon^{2R}), \\ \tilde{S}_\varepsilon^{R,N} &:= -\varepsilon^{N-R} (\varepsilon^{-N} t (\mathcal{L}_1^a(\varepsilon, v_\varepsilon), \varepsilon \mathcal{L}_2^a(\varepsilon, v_\varepsilon))) - \varepsilon^{R+M-2} (\tilde{v}_\varepsilon^{1R} \partial_\theta \tilde{v}_\varepsilon^R + \tilde{v}_\varepsilon^{2R} \partial_y \tilde{v}_\varepsilon^R) \\ &\quad - \frac{C \varepsilon^{2\nu-M-R-2}}{2} t (\partial_\theta, \varepsilon^2 \partial_y) (q_\varepsilon^a + \varepsilon^R \tilde{q}_\varepsilon^R)^2. \end{aligned}$$

The first (desired) effect is that the boxed term is desingularized into $t (0, \varepsilon^{-1} \partial_\theta h \tilde{v}_\varepsilon^{1R})$. Nevertheless, the desingularization of this hyperbolic singular part is at a cost on the parabolic part of the operator. The dissipation is transformed into

$$\mathcal{Q}_{\varepsilon, \lambda_\varepsilon} \tilde{v}_\varepsilon^R := \begin{pmatrix} \mathcal{Q}_{\varepsilon, \lambda_\varepsilon}^1 \tilde{v}_\varepsilon^R \\ \mathcal{Q}_{\varepsilon, \lambda_\varepsilon}^2 \tilde{v}_\varepsilon^R \end{pmatrix} = \frac{1}{\varepsilon^2} \begin{pmatrix} \mu (\partial_{\theta\theta} \tilde{v}_\varepsilon^{1R} + \varepsilon^2 \partial_{yy} \tilde{v}_\varepsilon^{1R}) + \lambda_\varepsilon (\partial_{\theta\theta} \tilde{v}_\varepsilon^{1R} + \partial_{yy} \tilde{v}_\varepsilon^{2R}) \\ \mu (\partial_{\theta\theta} \tilde{v}_\varepsilon^{2R} + \varepsilon^2 \partial_{yy} \tilde{v}_\varepsilon^{2R}) + \varepsilon^2 \lambda_\varepsilon (\partial_{\theta y} \tilde{v}_\varepsilon^{1R} + \partial_{yy} \tilde{v}_\varepsilon^{2R}) \end{pmatrix},$$

Yet the new dissipation $\mathcal{Q}_{\varepsilon, \lambda_\varepsilon}$ obtained might no longer satisfy (III.116). We prove, assuming λ_ε is small enough, that it does (see Lemma III.25). Thus, the control for bounded time is

a balance between the singular term and the dissipation. The parabolic approach described in the previous paragraph for the linearized equation can be considered again. It provides a L^2 -bound for the $(\tilde{q}_\varepsilon^l, \tilde{v}_\varepsilon^l)$ for time T_ε^l up to order 1 (see Proposition III.20).

In what follows we consider system (III.117*) for $(\tilde{q}_\varepsilon^R, \tilde{v}_\varepsilon^R)$ instead of system (III.111*). We perform energy estimates for (III.117*). This approach has a cost since when we go back to the initial variables $(q_\varepsilon^R, v_\varepsilon^R)$, we only get a bound for the family $\{\varepsilon v_\varepsilon^{2R}\}_\varepsilon$.

•**Non-linear terms**• There remains to check that non-linear terms such as

$$-\varepsilon^{R+M-2} (\tilde{v}_\varepsilon^{1R} \partial_\theta \tilde{q}_\varepsilon^R + \tilde{v}_\varepsilon^{2R} \partial_y \tilde{q}_\varepsilon^R) - C \varepsilon^{R+M-2} \tilde{q}_\varepsilon^R (\partial_\theta \tilde{v}_\varepsilon^{1R} + \partial_y \tilde{v}_\varepsilon^{2R}),$$

and

$$-\varepsilon^{R+M-2} (\tilde{v}_\varepsilon^{1R} \partial_\theta \tilde{v}_\varepsilon^R + \tilde{v}_\varepsilon^{2R} \partial_y \tilde{v}_\varepsilon^R) - \frac{C \varepsilon^{2\nu-M-R-2}}{2} {}^t(\partial_\theta, \varepsilon^2 \partial_y) (q_\varepsilon^a + \varepsilon^R \tilde{q}_\varepsilon^R)^2$$

contained in $(\tilde{S}_\varepsilon^{0,R,N}, \tilde{S}_\varepsilon^{R,N})$ do not prevent the control in L^2 norm for time of order 1. *A priori*, if R and ν are large enough, their contribution is weak. They will not interfere too much in the discussion.

Another aspect of the parabolic effect is the regularization of the term

$$\frac{C \varepsilon^{2\nu-M-R-2}}{2} {}^t(\partial_\theta, \varepsilon^2 \partial_y) (q_\varepsilon^a + \varepsilon^R \tilde{q}_\varepsilon^R)^2. \quad (\text{III.118})$$

Indeed if we multiply by the velocity and integrate, the derivative ${}^t(\partial_\theta, \varepsilon^2 \partial_y)$ can be passed to the velocity by an integration by parts. If ν is large enough, it decouples the equation on the pressure and the velocity. Similarly to the construction of the approximated solution $(q_\varepsilon^a, v_\varepsilon^a)$, we first look for estimates for the velocity by performing some non-symmetric energy method on Equation (III.117b), then we plug it into (III.117a) to deduce a control over the pressure.

The way to deal successively with the velocity and the pressure is reinforced by the fact that we only expect estimates for the pressure in anisotropic Sobolev spaces $H_{(1,\varepsilon)}^m$, whereas the velocity is estimated in classical Sobolev spaces. On the one hand, Equation (III.117a) is a hyperbolic equation with a singular transport $\varepsilon^{-1} h \partial_y$. We lose a power of ε each time we derivate with respect to θ . On the other hand, the same should occur for Equation (III.117b). However the viscosity absorbs the singular transport.

Nevertheless they are still coupled. The presence of the pressure in Equation (III.117b) through (III.118) is traduced on w_m (defined by Equation (III.13)) by a loss of precision with respect to the regularity m (see (III.13)).

We justify further those heuristical arguments by performing estimates for non-linear terms thanks to the Gagliardo-Nirenberg's inequality

Proposition III.19 (Gagliardo-Nirenberg's inequality). *Consider an integer $m \in \mathbb{N}$ and a multi-index α of length m ($|\alpha| = m$). There exists a positive constant C_g which only*

depends on m such that for any multi-index β such that $0 \leq \beta \leq \alpha$, for any functions $f, g \in H^m(\mathbb{T} \times \mathbb{R}) \cap L^\infty(\mathbb{T} \times \mathbb{R})$ we have

$$\left\| \partial^{\alpha-\beta} f \partial^\beta g \right\|_{L^2} \leq C_g \left(\|f\|_{\dot{H}^m} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{H}^m} \right).$$

The drawback of this inequality is that it requires that the function are bounded in L^∞ which is *a priori* not the case.

•Notations and results• To overcome the difficulty of L^∞ -estimates, we introduce the following characterized time $T_\varepsilon^* > 0$:

$$T_\varepsilon^* := \min \left(1, \sup_{T \in [0, T_\varepsilon]} \left\{ \forall t \in [0, T], \left\| \tilde{q}_\varepsilon^R(t, \cdot) \right\|_{H_{(1, \varepsilon)}^{m+3}(\mathbb{T} \times \mathbb{R})} \leq 2, \left\| \tilde{v}_\varepsilon^R(t, \cdot) \right\|_{H^{m+3}(\mathbb{T} \times \mathbb{R})} \leq 2 \right\} \right).$$

Thanks to Inequality (III.5), it provides L^∞ -estimates on the strip $[0, T_\varepsilon^*]$, for all $\varepsilon \in]0, 1]$:

$$\forall t \in [0, T_\varepsilon^*], \quad \left\| \sqrt{\varepsilon} \tilde{q}_\varepsilon^R(t, \cdot) \right\|_{W_{(1, \varepsilon)}^{m+1, \infty}(\mathbb{T} \times \mathbb{R})} \leq 2, \quad \left\| \tilde{v}_\varepsilon^R(t, \cdot) \right\|_{W^{m+1, \infty}(\mathbb{T} \times \mathbb{R})} \leq 2. \quad (\text{III.119})$$

The main result of this section is an accurate control of $(\tilde{q}_\varepsilon^R, \tilde{v}_\varepsilon^R)$ on the strip $[0, T_\varepsilon^*]$ in the following norm $H_{(1, \varepsilon)}^{m+3} \times H^{m+3}$.

Proposition III.20. *Select ν , M and m integers satisfying the condition (III.107). There exist ε_{crit} and K two positive constants such that:*

$$\forall \varepsilon \in]0, \varepsilon_{crit}], \quad \forall t \in [0, T_\varepsilon^*], \quad \left\| \tilde{q}_\varepsilon^R(t, \cdot) \right\|_{H_{(1, \varepsilon)}^{m+3}(\mathbb{T} \times \mathbb{R})}^2 \leq \varepsilon^{2w_m} K t, \quad (\text{III.120})$$

$$\left\| \tilde{v}_\varepsilon^R(t, \cdot) \right\|_{H^{m+3}(\mathbb{T} \times \mathbb{R})}^2 \leq \varepsilon^{2w_m} K t, \quad (\text{III.121})$$

where w_m is defined in Equation (III.13).

The proof of Proposition III.20 is done in two steps. First in Subsection 3.1, we prove the estimates for the velocity (III.120) taking profit of the dissipation. Then in Subsection 3.2 we inject the estimates obtained to deduce the control over the pressure (III.121).

Those estimates allow us to extend the solution further than the time T_ε^* on a time independent of ε . We deduce the minoration for the lifespan of time T_ε in Subsection 3.3.

To clarify the notations we drop the $\tilde{\cdot}$ in Sections 3.1 and 3.2.

3.1 Energy estimates for the velocity

In Section 3.1 we prove estimate (III.120) for the velocity in Proposition III.20. We crucially use that Equation (III.117b) is a parabolic type equation to prove an $L^2([0, T_\varepsilon^*], H^{m+4})$ estimate for v_ε^R . This gain of regularity is important in order to get a control in L^2 over the pressure.

Proposition III.21. *Let v_ε^R be a solution of (III.117b) on $[0, T_\varepsilon^*]$. There exist a positive constant ε_{crit} and two positive constants K_m^1 and K_m^2 (independent of ε) such that*

$$\forall \varepsilon \in]0, \varepsilon_{crit}], \quad \forall t \in [0, T_\varepsilon^*], \quad \|v_\varepsilon^R(t, \cdot)\|_{H^{m+3}(\mathbb{T} \times \mathbb{R})}^2 \leq \varepsilon^{2w_m} K_m^1 \left(e^{K_m^2 t} - 1 \right). \quad (\text{III.122})$$

Furthermore, select a multi-index $\alpha \in \mathbb{N}^2$ of length $|\alpha|$ smaller than $m + 4$; then

$$\forall \varepsilon \in]0, \varepsilon_{crit}], \quad \forall t \in [0, T_\varepsilon^*], \quad \int_0^t \left\| \varepsilon^{-(1-\delta_{\alpha_1, 0})} \partial^\alpha v_\varepsilon^R(s, \cdot) \right\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 ds \leq K_m^1 \varepsilon^{2w_m} t, \quad (\text{III.123})$$

where $\delta_{i,j}$ denotes the Kronecker symbol (of two integers): $\delta_{i,j} = 0$ if $i \neq j$ and $\delta_{i,i} = 1$. We recall that w_m satisfies Equation (III.13).

We prove Proposition III.21 by induction on the size of m settings:

$$\mathcal{P}(m) : \text{" Proposition III.21 holds up to the integer } m \text{ "}. \quad (\text{III.124})$$

To go from m to $m + 1$, the proof is based on an *energy method* for Equation (III.117b) in the homogeneous Sobolev space \mathring{H}^m .

Lemma III.22. *There exist ε_d and c_1 two positive constants such that for any $J \in \llbracket 0, m + 3 \rrbracket$ the following holds.*

i) *If $J = 0$, there exist two positive constants C_p and C_0^1 such that for all $\varepsilon \in]0, \varepsilon_d]$, and for all time $t \in [0, T_\varepsilon^*]$,*

$$\frac{1}{2} \partial_t \|v_\varepsilon^R(t, \cdot)\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 + c_1 \Phi_\varepsilon(\nabla, v_\varepsilon^R)(t) \leq C_0^1 \|v_\varepsilon^R(t, \cdot)\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 + C_p \varepsilon^{2w_m}. \quad (\text{III.125})$$

ii) *If $J \in \llbracket 1, m + 3 \rrbracket$, there exist four positive constants C_p , C_J^1 , C_J^2 and C_J^3 such that for all $\varepsilon \in]0, \varepsilon_d]$, and for all time $t \in [0, T_\varepsilon^*]$,*

$$\begin{aligned} \frac{1}{2} \partial_t \|v_\varepsilon^R(t, \cdot)\|_{H^J(\mathbb{T} \times \mathbb{R})}^2 + c_1 \sum_{|\alpha|=J} \Phi_\varepsilon(\nabla, \partial^\alpha v_\varepsilon^R)(t) &\leq C_J^1 \|v_\varepsilon^R(t, \cdot)\|_{H^J(\mathbb{T} \times \mathbb{R})}^2 + (J + 1) C_p \varepsilon^{2w_m} \\ &+ C_J^2 \|v_\varepsilon^R(t, \cdot)\|_{H^{J-1}(\mathbb{T} \times \mathbb{R})}^2 + C_J^3 \sum_{|\alpha|=J} \left\| \varepsilon^{-(1-\delta_{\alpha_1, 0})} \partial^\alpha v_\varepsilon^R(t, \cdot) \right\|_{L^2(\mathbb{T} \times \mathbb{R})}^2. \end{aligned}$$

First, we decompose Equation (III.117b) into

$$\partial_t v_\varepsilon^R + \mathcal{H} v_\varepsilon^R + \mathcal{A} v_\varepsilon^R + \mathcal{B} v_\varepsilon^R + \mathcal{C} v_\varepsilon^R - \mathcal{Q}_{\varepsilon, \lambda_\varepsilon} v_\varepsilon^R = S_\varepsilon^{R, N}, \quad (\text{III.126})$$

where the operators \mathcal{H} , \mathcal{A} , \mathcal{B} and \mathcal{C} are defined as follows:

$$\begin{aligned} \mathcal{H} v_\varepsilon^R &:= \varepsilon^{-1} h \partial_y v_\varepsilon^R, & \mathcal{A} v_\varepsilon^R &:= \varepsilon^{M-2} (v_\varepsilon^{1R} \partial_\theta + v_\varepsilon^{2R} \partial_y) {}^t(v_\varepsilon^{1a}, \varepsilon v_\varepsilon^{2a}), \\ \mathcal{C} v_\varepsilon^R &:= {}^t(0, \varepsilon^{-1} \partial_\theta h v_\varepsilon^{1R}), & \mathcal{B} v_\varepsilon^R &:= \varepsilon^{M-2} (v_\varepsilon^{1a} \partial_\theta v_\varepsilon^R + \varepsilon v_\varepsilon^{2a} \partial_y v_\varepsilon^R). \end{aligned}$$

We derivate (III.126) with respect to ∂^α , multiply by $\partial^\alpha v_\varepsilon^R$ and integrate:

$$\begin{aligned} \frac{1}{2} \partial_t \|\partial^\alpha v_\varepsilon^R\|_{L^2}^2 - \langle \partial^\alpha (\mathcal{Q}_{\varepsilon, \lambda_\varepsilon} v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle &= \langle \partial^\alpha S_\varepsilon^{R, N}, \partial^\alpha v_\varepsilon^R \rangle - \langle \partial^\alpha (\mathcal{A} v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle \\ &- \langle \partial^\alpha (\mathcal{B} v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle - \langle \partial^\alpha (\mathcal{C} v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle - \langle \partial^\alpha (\mathcal{H} v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle. \end{aligned} \quad (\text{III.127})$$

In Subection 3.1.1 we prove several lemmas in which we estimate each contribution in the above sum (III.127). We divide the study depending on the singularity (with respect to ε) of the different terms. We end Subsection 3.1.1 by proving Lemma III.22. Then in Subsection 3.1.2 we prove Proposition III.21.

3.1.1 Inequality of energy

First, we start by studying the two non-singular operators \mathcal{A} and \mathcal{B} . Their study is rather classical. We recall in both cases how we manage to exhibit an L^2 -estimate.

Then we move to the singular (with respect to ε) operators

$$\mathcal{H}f = \varepsilon^{-1} h \partial_y f \quad \text{and} \quad \mathcal{C}f = \varepsilon^{-1} {}^t(0, \partial_\theta h f).$$

They are also both singular with respect to the number of derivatives. This is not clear for operator \mathcal{C} . Nonetheless we perform an integration by parts to control it, making appear the derivative $\varepsilon^{-1} \partial_\theta$ as in (III.105). Such derivatives are controlled thanks to the dissipation $Q_{\varepsilon, \lambda_\varepsilon}$. Thus, we first prove estimate (III.116) for the dissipation and get a consequence for operator \mathcal{C} .

To deal with \mathcal{H} , the mechanism is more simple. Note that $\langle \mathcal{H}f, f \rangle \equiv 0$, so that the singularity vanishes. Nonetheless as soon as we derivate operators \mathcal{H} and \mathcal{C} with respect to ∂_θ , other singular contributions appear. Then, we deal with those extra terms (Lemmas III.27 and III.28).

Before proving Lemma III.22 at the end of Subsection 3.1.1 (and before Subsection 3.1.2) we estimate the term $S_\varepsilon^{R, N}$. One outcome on the proof performed is that the contribution of the pressure through the term (III.118) is small on $[0, T_\varepsilon^*]$ ($\forall \varepsilon \in]0, 1]$) when ν is assumed to be large enough.

•Non-singular Contributions• In this paragraph, we estimate the contribution of operators \mathcal{A} and \mathcal{B} in (III.127).

Lemma III.23. *Select a multi-index $\alpha \in \mathbb{N}^2$ of length smaller than $m + 3$. Then,*

i) if $|\alpha| = 0$, there exists $C_{\mathcal{A}}^1$ a positive constant such that for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^]$:*

$$|\langle \mathcal{A}v_\varepsilon^R, v_\varepsilon^R \rangle| (t) \leq C_{\mathcal{A}}^1 \|v_\varepsilon^R(t, \cdot)\|_{L^2(\mathbb{T} \times \mathbb{R})}^2; \quad (\text{III.128})$$

ii) if $|\alpha| \geq 1$ there exist $C_{\mathcal{A}}^1$ and $C_{\mathcal{A}}^2$ two positive constants such that for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^]$:*

$$|\langle \partial^\alpha (\mathcal{A}v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle| (t) \leq C_{\mathcal{A}}^1 \|v_\varepsilon^R(t, \cdot)\|_{H^{|\alpha|}(\mathbb{T} \times \mathbb{R})}^2 + C_{\mathcal{A}}^2 \|v_\varepsilon^R(t, \cdot)\|_{H^{|\alpha|-1}(\mathbb{T} \times \mathbb{R})}^2.$$

Proof of Lemma III.23. Select a multi-index α such that $|\alpha| \leq m + 3$. We only deal with the case $|\alpha| \geq 1$. The proof in the case $|\alpha| = 0$ is the same. Applying the Cauchy-Schwarz

inequality:

$$\begin{aligned} |\langle \partial^\alpha (\mathcal{A}v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle| &= \varepsilon^{M-2} |\langle \partial^\alpha ((v_\varepsilon^{1R} \partial_\theta + v_\varepsilon^{2R} \partial_y)^t (v_\varepsilon^{1a}, \varepsilon v_\varepsilon^{2a})), \partial^\alpha v_\varepsilon^R \rangle|, \\ &\leq \frac{1}{2} \left(\|\partial^\alpha ((v_\varepsilon^{1R} \partial_\theta + v_\varepsilon^{2R} \partial_y)^t (v_\varepsilon^{1a}, \varepsilon v_\varepsilon^{2a}))\|_{L^2}^2 + \|\partial^\alpha v_\varepsilon^R\|_{L^2}^2 \right), \\ &\leq \|\partial^\alpha (v_\varepsilon^{1R} \partial_\theta^t (v_\varepsilon^{1a}, \varepsilon v_\varepsilon^{2a}))\|_{L^2}^2 + \|\partial^\alpha (v_\varepsilon^{2R} \partial_y^t (v_\varepsilon^{1a}, \varepsilon v_\varepsilon^{2a}))\|_{L^2}^2 + \frac{1}{2} \|\partial^\alpha v_\varepsilon^R\|_{L^2}^2. \end{aligned}$$

◦ We start by estimating $\|\partial^\alpha (v_\varepsilon^{1R} \partial_\theta^t (v_\varepsilon^{1a}, \varepsilon v_\varepsilon^{2a}))\|_{L^2}^2$. We apply the Leibniz formula (for the derivative of a product). There exists $\{C_{\alpha,\beta}\}_\beta$ a family of positive constants such that:

$$\partial^\alpha (v_\varepsilon^{1R} \partial_\theta^t (v_\varepsilon^{1a}, \varepsilon v_\varepsilon^{2a})) = \sum_{0 \leq \beta \leq \alpha} C_{\alpha,\beta} \partial^{\alpha-\beta} v_\varepsilon^{1R} \partial^\beta \partial_\theta^t (v_\varepsilon^{1a}, \varepsilon v_\varepsilon^{2a}).$$

Apply the Minkovski inequality, then use (III.108) together with the classical embedding $H^{m+6} \hookrightarrow W^{m+4,\infty}$:

$$\begin{aligned} \|\partial^\alpha (v_\varepsilon^{1R} \partial_\theta^t (v_\varepsilon^{1a}, \varepsilon v_\varepsilon^{2a}))\|_{L^2} &\leq \sum_{0 \leq \beta \leq \alpha} C_{\alpha,\beta} \|\partial^{\alpha-\beta} v_\varepsilon^{1R} \partial^\beta \partial_\theta^t (v_\varepsilon^{1a}, \varepsilon v_\varepsilon^{2a})\|_{L^2}, \\ &\leq \sum_{0 \leq \beta \leq \alpha} C_{\alpha,\beta} C_\alpha \|\partial^\beta v_\varepsilon^R\|_{L^2}. \end{aligned}$$

Sorting the term contingent on the number of derivatives acting on v_ε^R , there exists a positive constant $C_\alpha > 0$ such that for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$,

$$\|\partial^\alpha (v_\varepsilon^{1R} \partial_\theta^t (v_\varepsilon^{1a}, \varepsilon v_\varepsilon^{2a}))\|_{L^2}^2(t) \leq C_\alpha \left(\|v_\varepsilon^R(t, \cdot)\|_{H^{|\alpha|-1}}^2 + \|v_\varepsilon^R(t, \cdot)\|_{H^{|\alpha|}}^2 \right).$$

◦ We can perform a duplicate study for $v_\varepsilon^{2R} \partial_y^t (v_\varepsilon^{1a}, \varepsilon v_\varepsilon^{2a})$. Sorting all the terms we obtain, for all $\varepsilon \in]0, 1]$, for all time $t \in [0, T_\varepsilon^*]$,

$$|\langle \partial^\alpha (\mathcal{A}v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle|(t) \leq 2C_\alpha \|v_\varepsilon^R(t, \cdot)\|_{H^{|\alpha|-1}(\mathbb{T} \times \mathbb{R})}^2 + \left(2C_\alpha + \frac{1}{2}\right) \|v_\varepsilon^R(t, \cdot)\|_{H^{|\alpha|}(\mathbb{T} \times \mathbb{R})}^2.$$

This terminates the proof. ■

The main difference between \mathcal{B} and \mathcal{A} is that \mathcal{B} contains some derivatives of order one acting on v_ε^R , such as $v_\varepsilon^{1a} \partial_\theta v_\varepsilon^R$. We can apply the same study for operators \mathcal{B} . However, terms of the form $\|v_\varepsilon^R\|_{H^{|\alpha|+1}}^\circ$ appear in the final estimate. As we expect the dissipation $Q_{\varepsilon, \lambda_\varepsilon}$ to absorb it, the final estimations (III.125)-(III.126) still work. However in concern of being accurate, we actually prove that it only depends on $\|v_\varepsilon^R\|_{H^{|\alpha|}}$, not on the term $\|v_\varepsilon^R\|_{H^{|\alpha|+1}}^\circ$. To do so we notice that integrating by parts,

$$\langle v_\varepsilon^{1a} \partial_\theta v_\varepsilon^R, v_\varepsilon^R \rangle = - \int_{\mathbb{T} \times \mathbb{R}} \partial_\theta v_\varepsilon^{1a} \frac{1}{2} |v_\varepsilon^R|^2 d\theta dy. \quad (\text{III.129})$$

We can pass the derivative ∂_θ on the approximated solution. We prove:

Lemma III.24. *Select a multi-index $\alpha \in \mathbb{N}^2$ of length smaller than $m + 3$. Then,*

i) *if $|\alpha| = 0$, there exists $C_{\mathcal{B}}^1$ a positive constant such that for all $\varepsilon \in]0, 1]$, for all time $t \in [0, T_\varepsilon^*]$,*

$$|\langle \partial^\alpha (\mathcal{B}v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle| (t) \leq C_{\mathcal{B}}^1 \|v_\varepsilon^R(t, \cdot)\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 ;$$

ii) *if $|\alpha| \geq 1$, there exist $C_{\mathcal{B}}^1$ and $C_{\mathcal{B}}^2$ two positive constants such that for all $\varepsilon \in]0, 1]$, for all time $t \in [0, T_\varepsilon^*]$,*

$$|\langle \partial^\alpha (\mathcal{B}v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle| (t) \leq C_{\mathcal{B}}^1 \|v_\varepsilon^R(t, \cdot)\|_{H^{|\alpha|}(\mathbb{T} \times \mathbb{R})}^2 + C_{\mathcal{B}}^2 \|v_\varepsilon^R(t, \cdot)\|_{H^{|\alpha|-1}(\mathbb{T} \times \mathbb{R})}^2 .$$

Proof of Lemma III.24. Choose a multi-index α such that $|\alpha| \leq m + 3$. We only make the proof in the case $|\alpha| \geq 1$. The proof in the case $|\alpha| = 0$ is essentially done using the above remark (III.129). First

$$\begin{aligned} \langle \partial^\alpha (\mathcal{B}v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle &= \varepsilon^{M-2} \langle \partial^\alpha (v_\varepsilon^{1a} \partial_\theta v_\varepsilon^R + \varepsilon v_\varepsilon^{2a} \partial_y v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle, \\ &= \varepsilon^{M-2} \langle \partial^\alpha (v_\varepsilon^{1a} \partial_\theta v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle + \varepsilon^{M-2} \langle \partial^\alpha (\varepsilon v_\varepsilon^{2a} \partial_y v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle. \end{aligned}$$

◦ We start by estimating $\langle \partial^\alpha (v_\varepsilon^{1a} \partial_\theta v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle$. We make the structure found above (III.129) appear. Applying the Leibniz formula, there exists $\{C_{\alpha,\beta}\}_\beta$ a family of positive constants such that:

$$\partial^\alpha (v_\varepsilon^{1a} \partial_\theta v_\varepsilon^R) = \sum_{0 \leq \beta < \alpha} C_{\alpha,\beta} \partial^{\alpha-\beta} v_\varepsilon^{1a} \partial^\beta \partial_\theta v_\varepsilon^R = \sum_{0 \leq \beta < \alpha} C_{\alpha,\beta} \partial^{\alpha-\beta} v_\varepsilon^{1a} \partial^\beta \partial_\theta v_\varepsilon^R + v_\varepsilon^{1a} \partial^\alpha \partial_\theta v_\varepsilon^R.$$

Multiplying by $\partial^\alpha v_\varepsilon^R$ and integrating we obtain:

$$\begin{aligned} \langle \partial^\alpha (v_\varepsilon^{1a} \partial_\theta v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle &= \sum_{0 \leq \beta < \alpha} C_{\alpha,\beta} \left\langle \partial^{\alpha-\beta} v_\varepsilon^{1a} \partial^\beta \partial_\theta v_\varepsilon^R, \partial^\alpha v_\varepsilon^R \right\rangle + \int_{\mathbb{T} \times \mathbb{R}} v_\varepsilon^{1a} \frac{1}{2} \partial_\theta (\partial^\alpha v_\varepsilon^R)^2 d\theta dy, \\ &= \sum_{0 \leq \beta < \alpha} C_{\alpha,\beta} \left\langle \partial^{\alpha-\beta} v_\varepsilon^{1a} \partial^\beta \partial_\theta v_\varepsilon^R, \partial^\alpha v_\varepsilon^R \right\rangle - \int_{\mathbb{T} \times \mathbb{R}} \partial_\theta v_\varepsilon^{1a} (\partial^\alpha v_\varepsilon^R)^2 d\theta dy. \end{aligned}$$

First we have:

$$\left| \left\langle \partial^{\alpha-\beta} v_\varepsilon^{1a} \partial^\beta \partial_\theta v_\varepsilon^R, \partial^\alpha v_\varepsilon^R \right\rangle \right| \leq \frac{1}{2} \left(\left\| \partial^{\alpha-\beta} v_\varepsilon^{1a} \partial^\beta \partial_\theta v_\varepsilon^R \right\|_{L^2}^2 + \left\| \partial^\alpha v_\varepsilon^R \right\|_{L^2}^2 \right) .$$

Then apply the Minkovski inequality together with (III.108) and the injection $H^{m+6} \hookrightarrow W^{m+4,\infty}$ to get

$$\begin{aligned} |\langle \partial^\alpha (v_\varepsilon^{1a} \partial_\theta v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle| &\leq \sum_{0 \leq \beta < \alpha} \frac{C_{\alpha,\beta}}{2} \left(\left\| \partial^{\alpha-\beta} v_\varepsilon^{1a} \partial^\beta \partial_\theta v_\varepsilon^R \right\|_{L^2}^2 + \left\| \partial^\alpha v_\varepsilon^R \right\|_{L^2}^2 \right) \\ &\quad + \int_{\mathbb{T} \times \mathbb{R}} |\partial_\theta v_\varepsilon^{1a}| |\partial^\alpha v_\varepsilon^R|^2 d\theta dy \\ &\leq \sum_{0 \leq \beta < \alpha} \frac{C_{\alpha,\beta}(C_a + 1)}{2} \left(\left\| \partial^\beta \partial_\theta v_\varepsilon^R \right\|_{L^2}^2 + \left\| \partial^\alpha v_\varepsilon^R \right\|_{L^2}^2 \right) + C_a \left\| \partial^\alpha v_\varepsilon^R \right\|_{L^2}^2 . \end{aligned}$$

In the sum the derivatives are of order at most $|\alpha|$. Then sorting the terms out depending on the number of derivatives acting on v_ε^R , there exists a positive constant $C_\alpha > 0$ such that for all $\varepsilon \in]0, 1]$, for all $t \in [0, T_\varepsilon^*]$,

$$\langle \partial^\alpha (v_\varepsilon^{1a} \partial_\theta v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle(t) \leq C_\alpha \left(\|v_\varepsilon^R(t, \cdot)\|_{H^{|\alpha|-1}(\mathbb{T} \times \mathbb{R})}^2 + \|v_\varepsilon^R(t, \cdot)\|_{H^{|\alpha|}(\mathbb{T} \times \mathbb{R})}^2 \right).$$

◦ We can perform the same study for $\langle \partial^\alpha (\varepsilon v_\varepsilon^{2a} \partial_y v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle$. Sorting all the terms out we obtain for all $\varepsilon \in]0, 1]$, for all $t \in [0, T_\varepsilon^*]$,

$$|\langle \partial^\alpha (\mathcal{B} v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle|(t) \leq 2C_\alpha \|v_\varepsilon^R(t, \cdot)\|_{H^{|\alpha|-1}(\mathbb{T} \times \mathbb{R})}^2 + 2C_\alpha \|v_\varepsilon^R(t, \cdot)\|_{H^{|\alpha|}(\mathbb{T} \times \mathbb{R})}^2,$$

for some $C_\alpha > 0$. This completes the proof. \blacksquare

•Estimate for the dissipation - Consequence• We now estimate the term involving the dissipation $Q_{\varepsilon, \lambda_\varepsilon}$.

Lemma III.25. *Let $\delta > 0$. Select a constant λ which satisfies*

$$\lambda < 4\mu. \quad (\text{III.130})$$

Then there exists $\varepsilon_d \in]0, 1]$, such that for all $\varepsilon \in]0, \varepsilon_d]$, $Q_{\varepsilon, \lambda_\varepsilon}$ satisfies Inequality (III.116). That is to say there exists $c_0 > 0$ (independent of ε) such that for any function $f \in H^1(\mathbb{T} \times \mathbb{R}, \mathbb{R}^2)$:

$$\forall \varepsilon \in]0, \varepsilon_d], \quad -\langle Q_{\varepsilon, \lambda_\varepsilon} f, f \rangle \geq c_0 \Phi_\varepsilon(\nabla, f).$$

Proof of Lemma III.25. Let $f \in H^1(\mathbb{T} \times \mathbb{R}, \mathbb{R}^2)$. We decompose f in Fourier series (in θ) into:

$$f(\theta, y) := \sum_{k \in \mathbb{Z}} f_k(y) e^{ik\theta}.$$

Applying a discrete Fourier transform in the variable θ and a Fourier transform in the variable y , we interpret $\langle Q_{\varepsilon, \lambda_\varepsilon} f, f \rangle$ as a quadratic form in the variables \widehat{f}_k^1 and \widehat{f}_k^2 . By the Parseval equality, up to a constant (that we forget here),

$$-\langle Q_{\varepsilon, \lambda_\varepsilon} f, f \rangle = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \widehat{f}_k(\xi) Q_{\varepsilon, \lambda_\varepsilon}(k, \xi) \widehat{f}_k(\xi) d\xi,$$

where $Q_{\varepsilon, \lambda_\varepsilon}$ is defined as:

$$Q_{\varepsilon, \lambda_\varepsilon}(k, \xi) := \begin{pmatrix} (\mu + \lambda_\varepsilon) \frac{k^2}{\varepsilon^2} + \xi^2 & \frac{\lambda_\varepsilon}{2} \left(1 + \frac{1}{\varepsilon^2}\right) k \xi \\ \frac{\lambda_\varepsilon}{2} \left(1 + \frac{1}{\varepsilon^2}\right) k \xi & \mu \frac{k^2}{\varepsilon^2} + (\mu + \lambda_\varepsilon) \xi^2 \end{pmatrix}, \quad k \in \mathbb{Z}, \quad \xi \in \mathbb{R}.$$

In this way, to prove Inequality (III.116) we show that there exist ε_d and c_0 two positive constants such that for all $g \in H^1(\mathbb{R}, \mathbb{R}^2)$:

$$\forall k \in \mathbb{Z}, \quad \forall \xi \in \mathbb{R}, \quad \widehat{g}(\xi) Q_{\varepsilon, \lambda_\varepsilon}(k, \xi) \widehat{g}(\xi) \geq c_0 (\varepsilon^{-2} k^2 + \xi^2) (\widehat{g}_1^2(\xi) + \widehat{g}_2^2(\xi)). \quad (\text{III.131})$$

At fixed $(k, \xi) \in \mathbb{Z} \times \mathbb{R}$, $Q_{\varepsilon, \lambda_\varepsilon}(k, \xi)$ is a real symmetric matrix, it is diagonalisable. We compute its eigenvalues:

$$\begin{aligned}\mu_\varepsilon^1(k, \xi) &:= \frac{2\mu + \lambda_\varepsilon}{2}((\varepsilon^{-1}k)^2 + \xi^2) + \frac{\lambda_\varepsilon}{2} \sqrt{((\varepsilon^{-1}k)^4 + \xi^4) + (\varepsilon(\varepsilon^{-1}k)\xi)^2 + \varepsilon^{-2}(\varepsilon^{-1}k\xi)^2}, \\ \mu_\varepsilon^2(k, \xi) &:= \frac{2\mu + \lambda_\varepsilon}{2}((\varepsilon^{-1}k)^2 + \xi^2) - \frac{\lambda_\varepsilon}{2} \sqrt{((\varepsilon^{-1}k)^4 + \xi^4) + (\varepsilon(\varepsilon^{-1}k)\xi)^2 + \varepsilon^{-2}(\varepsilon^{-1}k\xi)^2}.\end{aligned}$$

In the end of the proof, we show that for any $(k, \xi) \in \mathbb{Z} \times \mathbb{R}$ we have

$$\mu_\varepsilon^2(k, \xi) \geq c_0 (\varepsilon^{-2} k^2 + \xi^2) (\widehat{g}_1^2(\xi) + \widehat{g}_2^2(\xi)).$$

Since $\mu_\varepsilon^1 \geq \mu_\varepsilon^2$, we clearly obtain Inequality (III.131).

◦ We define the function $\mu_\varepsilon : \mathbb{R}^2 \rightarrow \mathbb{R}$ by:

$$\mu_\varepsilon(x, y) := \frac{2\mu + \lambda_\varepsilon}{2}(x^2 + y^2) - \frac{\lambda_\varepsilon}{2} \sqrt{(x^4 + y^4) + (\varepsilon xy)^2 + \varepsilon^{-2}(xy)^2}.$$

Then there exist ε_d and c_0 (independent of ε) such that

$$\forall (x, y) \in \mathbb{R}^2, \quad \mu_\varepsilon(x, y) \geq c_0(x^2 + y^2). \quad (\text{III.132})$$

-First of all the function μ_ε is homogeneous of order 2 in the sense that it satisfies for all $\alpha \in \mathbb{R}$:

$$\forall (x, y) \in \mathbb{R} \times \mathbb{R}, \quad \mu_\varepsilon(\alpha x, \alpha y) = \alpha^2 \mu_\varepsilon(x, y).$$

Thus, we prove (III.132) on the restricted set $(x, y) \in \mathbb{S}^1$ the sphere of center 0 and radius 1.

-We expand μ_ε as ε goes to 0^+ :

$$\mu_\varepsilon(x, y) = (x^2 + y^2) - \frac{\lambda_\varepsilon}{2\varepsilon} |x||y| + O(\varepsilon),$$

where $O(\varepsilon)$ is uniform in $(x, y) \in \mathbb{S}^1$. Let us recall that we have:

$$\forall (x, y) \in \mathbb{R}^2, \quad (x^2 + y^2) - c|x||y| > 0 \quad \Longleftrightarrow \quad c < 2.$$

That is the case if and only if $\lambda < 4\mu$, *i.e.* assumption (III.130) is satisfied. We get an uniform bound in ε of μ_ε (in ε). Finally, there exists ε_d and c_0 two positive constants such that Inequality (III.132) is satisfied.

◦ Plugging $x = \varepsilon^{-1}k$ and $y = \xi$ in Inequality (III.132), we obtain for all $\varepsilon \in]0, \varepsilon_d]$:

$$\forall (k, \xi) \in \mathbb{Z} \times \mathbb{R}, \quad \mu_\varepsilon^2(k, \xi) \geq c_0 ((\varepsilon^{-1}k)^2 + \xi^2).$$

We deduce that

$$\widehat{f}_k(\xi) Q_{\varepsilon, \lambda_\varepsilon}(k, \xi) \widehat{f}_k(\xi) \geq c_0 ((\varepsilon^{-1}k)^2 + \xi^2) \|\widehat{f}_k(\xi)\|^2.$$

Summing over $k \in \mathbb{Z}$ and integrating with respect to ξ , we have:

$$-\langle Q_{\varepsilon, \lambda_\varepsilon} f, f \rangle \geq c_0 \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} ((\varepsilon^{-1}k)^2 + \xi^2) \|\widehat{f}_k(\xi)\|^2 d\xi = c_0 \Phi_\varepsilon(\nabla, f).$$

■

Remark. The hypothesis $\lambda_\varepsilon := \lambda \varepsilon$ is not necessary to get Inequality (III.116). What is important is that λ_ε decreases fast enough to 0. For instance that λ_ε satisfies

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^{-1} \lambda_\varepsilon < 4\mu. \quad (\text{III.133})$$

If $\lambda_\varepsilon \not\prec 4\mu\varepsilon$, Inequality (III.132) is no longer satisfied. Choosing $(x, y) = (1, 1)$:

$$\mu_\varepsilon(1, 1) = (2\mu + \lambda_\varepsilon) - \frac{\lambda_\varepsilon}{2} \sqrt{2 + \varepsilon^2 + \frac{1}{\varepsilon^2}}.$$

For ε small enough it becomes negative. Nonetheless, it does not mean that $\mu_\varepsilon^2(1, 1) < 0$ for ε small. We can hope for Inequality (III.116) assuming a less restrictive assumption than (III.130).

Having Inequality (III.116) in mind, we absorb some singular contribution of operator \mathcal{C} .

Corollary III.26 (Absorption of \mathcal{C}). *Select a positive constant c_1 and a multi-index $\alpha \in \mathbb{N}^2$ of length less than $m + 3$. Then for all $\varepsilon \in]0, 1]$ for all time $t \in [0, T_\varepsilon^*]$:*

$$-\frac{c_1 \|h\|_{L^\infty(\mathbb{T})}}{2} \Phi_\varepsilon(\nabla, \partial^\alpha v_\varepsilon^R(t, \cdot)) + \langle \mathcal{C} \partial^\alpha v_\varepsilon^R(t, \cdot), \partial^\alpha v_\varepsilon^R(t, \cdot) \rangle \leq \frac{\|h\|_{L^\infty(\mathbb{T})}}{2c_1} \|\partial^\alpha v_\varepsilon^R(t, \cdot)\|_{L^2(\mathbb{T} \times \mathbb{R})}^2.$$

Proof of Corollary III.26. We prove the result in the particular case $\alpha = (0, 0)$. We select c_1 a positive constant. The idea is to integrate by parts with respect to the variable θ to make the weighted derivative $\varepsilon^{-1} \partial_\theta$ appear as performed in (III.105).

$$\begin{aligned} |\langle \mathcal{C} v_\varepsilon^R, v_\varepsilon^R \rangle| &= \left| \varepsilon^{-1} \int \partial_\theta h v_\varepsilon^{1R} v_\varepsilon^{2R} d\theta dy \right|, \\ &= \left| \int h \varepsilon^{-1} \partial_\theta v_\varepsilon^{1R} v_\varepsilon^{2R} d\theta dy + \int h v_\varepsilon^{1R} \varepsilon^{-1} \partial_\theta v_\varepsilon^{2R} d\theta dy \right|, \\ &\leq \frac{1}{2} \|h\|_{L^\infty} \left(c_1^{-1} \|v_\varepsilon^{1R}\|_{L^2}^2 + c_1^{-1} \|v_\varepsilon^{2R}\|_{L^2}^2 + c_1 \|\varepsilon^{-1} \partial_\theta v_\varepsilon^{1R}\|_{L^2}^2 + c_1 \|\varepsilon^{-1} \partial_\theta v_\varepsilon^{2R}\|_{L^2}^2 \right), \\ &\leq \frac{1}{2c_1} \|h\|_{L^\infty} \|v_\varepsilon^R\|_{L^2}^2 + \frac{c_1}{2} \|h\|_{L^\infty} \Phi_\varepsilon(\nabla, v_\varepsilon^R). \end{aligned}$$

Replacing v_ε^R by $\partial^\alpha v_\varepsilon^R$, this proves the result. ■

•**Singular Contributions**• Presently, we study the singular operators \mathcal{H} and \mathcal{C} . We decompose their action into

$$\begin{aligned} \langle \partial^\alpha (\mathcal{H} v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle &= \langle \mathcal{H} (\partial^\alpha v_\varepsilon^R), v_\varepsilon^R \rangle + \langle [\partial^\alpha, \mathcal{H}] v_\varepsilon^R, \partial^\alpha v_\varepsilon^R \rangle, \\ \langle \partial^\alpha (\mathcal{C} v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle &= \langle \mathcal{C} (\partial^\alpha v_\varepsilon^R), v_\varepsilon^R \rangle + \langle [\partial^\alpha, \mathcal{C}] v_\varepsilon^R, \partial^\alpha v_\varepsilon^R \rangle. \end{aligned}$$

by making some commutators appear. Since $\langle \mathcal{H} \partial^\alpha v_\varepsilon^R, v_\varepsilon^R \rangle = 0$ and we deal with $\langle \mathcal{C} \partial^\alpha v_\varepsilon^R, \partial^\alpha v_\varepsilon^R \rangle$ thanks to Lemma III.26, there remains to study those two commutators. Each time we assume $|\alpha| \geq 1$ so that the commutators do not vanish.

Lemma III.27. *Select a multi-index $\alpha \in \mathbb{N}^2$ with length satisfying $1 \leq |\alpha| \leq m + 3$. There exist $C_{\mathcal{H}}^1$, $C_{\mathcal{H}}^2$ and $C_{\mathcal{H}}^3$ three positive constants such that for all $\varepsilon \in]0, 1]$, for all time $t \in [0, T_\varepsilon^*]$,*

$$\begin{aligned} |\langle [\partial^\alpha, \mathcal{H}] v_\varepsilon^R, \partial^\alpha v_\varepsilon^R \rangle| (t) &\leq C_{\mathcal{H}}^1 \|v_\varepsilon^R(t, \cdot)\|_{H^{|\alpha|}(\mathbb{T} \times \mathbb{R})}^2 + C_{\mathcal{H}}^2 \|v_\varepsilon^R(t, \cdot)\|_{H^{|\alpha|-1}(\mathbb{T} \times \mathbb{R})}^2 \\ &\quad + C_{\mathcal{H}}^3 \left\| \varepsilon^{-(1-\delta_{\alpha_1, 0})} \partial^\alpha v_\varepsilon^R(t, \cdot) \right\|_{L^2(\mathbb{T} \times \mathbb{R})}^2. \end{aligned}$$

Proof of Lemma III.27. Select a multi-index $\alpha \in \mathbb{N}^2$ such that $1 \leq |\alpha| \leq m + 3$.

◦ First, assume $\alpha_1 = 0$. Then the result is obvious because ∂^α and \mathcal{H} commute : $[\partial^\alpha, \mathcal{H}] = 0$.

◦ We assume $\alpha_1 > 0$. \mathcal{H} is singular with respect to ε . Since we expect to control the weighted derivative $\varepsilon^{-1} \partial_\theta$, we deal with the singularity putting ε^{-1} and $\partial^\alpha v_\varepsilon^R$ together.

$$|\langle [\partial^\alpha, \mathcal{H}] v_\varepsilon^R, \partial^\alpha v_\varepsilon^R \rangle| = |\langle [\partial^\alpha, \varepsilon \mathcal{H}] v_\varepsilon^R, \varepsilon^{-1} \partial^\alpha v_\varepsilon^R \rangle| \leq \frac{1}{2} \left(\|[\partial^\alpha, h \partial_y] v_\varepsilon^R\|_{L^2}^2 + \|\varepsilon^{-1} \partial^\alpha v_\varepsilon^R\|_{L^2}^2 \right)$$

The operator $[\partial^\alpha, h \partial_y]$ is no longer singular and is of order $|\alpha|$,

$$\|[\partial^\alpha, h \partial_y] v_\varepsilon^R\|_{L^2}^2 \leq C_h \|v_\varepsilon^R\|_{H^{|\alpha|}}^2,$$

for some $C_h \in \mathbb{R}_+^*$. Finally decomposing the Sobolev norm as $\|\cdot\|_{H^{|\alpha|}}^2 = \|\cdot\|_{H^{|\alpha|-1}}^2 + \|\cdot\|_{H^{|\alpha|}}^2$, we obtain for all $\varepsilon \in [0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$,

$$\begin{aligned} |\langle [\partial^\alpha, \mathcal{H}] v_\varepsilon^R, \partial^\alpha v_\varepsilon^R \rangle| (t) &\leq \frac{1}{2} C_h \|v_\varepsilon^R(t, \cdot)\|_{H^{|\alpha|-1}(\mathbb{T} \times \mathbb{R})}^2 \\ &\quad + \frac{1}{2} C_h \|v_\varepsilon^R(t, \cdot)\|_{H^{|\alpha|}(\mathbb{T} \times \mathbb{R})}^2 + \frac{1}{2} \left\| \varepsilon^{-1} \partial^\alpha v_\varepsilon^R(t, \cdot) \right\|_{L^2(\mathbb{T} \times \mathbb{R})}^2. \end{aligned}$$

■

Eventually the contribution of the commutator between \mathcal{C} and ∂^α is estimated in like manner:

Lemma III.28. *Choose a multi-index $\alpha \in \mathbb{N}^2$ with length satisfying $1 \leq |\alpha| \leq m + 3$. There exist $C_{\mathcal{C}}^1$, $C_{\mathcal{C}}^2$ and $C_{\mathcal{C}}^3$ three positive constants such that for all $\varepsilon \in]0, 1]$, for all time $t \in [0, T_\varepsilon^*]$,*

$$\begin{aligned} |\langle [\partial^\alpha, \mathcal{C}] v_\varepsilon^R, \partial^\alpha v_\varepsilon^R \rangle| (t) &\leq C_{\mathcal{C}}^1 \|v_\varepsilon^R(t, \cdot)\|_{H^{|\alpha|}(\mathbb{T} \times \mathbb{R})}^2 + C_{\mathcal{C}}^2 \|v_\varepsilon^R(t, \cdot)\|_{H^{|\alpha|-1}(\mathbb{T} \times \mathbb{R})}^2 \\ &\quad + C_{\mathcal{C}}^3 \left\| \varepsilon^{-(1-\delta_{\alpha_1, 0})} \partial^\alpha v_\varepsilon^R(t, \cdot) \right\|_{L^2(\mathbb{T} \times \mathbb{R})}^2. \end{aligned}$$

• **Estimate of the source $S_\varepsilon^{R, N}$** • There remains to get a control over the term $S_\varepsilon^{R, N}$. It is in some way the most difficult contribution to deal with. It contains the non-linear contribution of $v \cdot \nabla v$, together with $q \nabla q$.

The contribution of the term of the form $v \cdot \nabla v$ can not be dealt with an integration by parts to decrease the number of derivatives as performed in (III.105). However, we can absorb it thanks to the dissipation.

The term $q\nabla q$ is more problematic. Indeed, performing a L^2 -estimate for this contribution, it brings terms of the form $\|q\|_{H^1}$. We need an H^1 -estimate on the pressure. This is not possible. We still overcome this difficulty by an integration by parts:

$$\int_{\mathbb{T} \times \mathbb{R}} q \partial_{\theta} q v_{\varepsilon}^R d\theta dy = - \int_{\mathbb{T} \times \mathbb{R}} \partial_{\theta} v_{\varepsilon}^R \frac{1}{2} |q|^2 d\theta dy.$$

In other words, we prefer the derivative to act on the velocity instead of the pressure. We can state the following lemma.

Lemma III.29. *[Control over $S_{\varepsilon}^{R,N}$] Select a multi-index $\alpha \in \mathbb{N}^2$ with length smaller than $m+3$ and a positive constant C_S . There exist C_S^1 , C_p two positive constants such that for any $\varepsilon \in]0, 1]$, for any time $t \in [0, T_{\varepsilon}^*]$,*

$$|\langle \partial^{\alpha} S_{\varepsilon}^{R,N}, \partial^{\alpha} v_{\varepsilon}^R \rangle|(t, \cdot) \leq C_S \|v_{\varepsilon}^R(t, \cdot)\|_{H^{|\alpha|+1}(\mathbb{T} \times \mathbb{R})}^2 + C_S^1 \|v_{\varepsilon}^R(t, \cdot)\|_{H^{|\alpha|}(\mathbb{T} \times \mathbb{R})}^2 + C_p \varepsilon^{2w_m}.$$

Proof of Lemma III.29. Consider $\alpha \in \mathbb{N}^2$ satisfying $|\alpha| \leq m+3$. We decompose the source into

$$\begin{aligned} \langle \partial^{\alpha} S_{\varepsilon}^{R,N}, \partial^{\alpha} v_{\varepsilon}^R \rangle &= -\varepsilon^{N-R} \langle \partial^{\alpha} (\varepsilon^{-N} \mathcal{L}^a(\varepsilon, v_{\varepsilon}^a)), \partial^{\alpha} v_{\varepsilon}^R \rangle \\ &\quad - \varepsilon^{R+M-2} \langle \partial^{\alpha} (v_{\varepsilon}^{1R} \partial_{\theta} v_{\varepsilon}^R + \varepsilon v_{\varepsilon}^{2R} \partial_y v_{\varepsilon}^R), \partial^{\alpha} v_{\varepsilon}^R \rangle \\ &\quad - C 2^{-1} \varepsilon^{2\nu-M-R-2} \langle {}^t(\partial_{\theta}, \varepsilon \partial_y) \partial^{\alpha} (q_{\varepsilon}^a + \varepsilon^R q_{\varepsilon}^R)^2, \partial^{\alpha} v_{\varepsilon}^R \rangle. \end{aligned}$$

We estimate each term.

◦ *The first term: contribution of the approximated solution.* v_{ε}^a is an approximated solution for the operator \mathcal{L}^a . According to Proposition III.1, it satisfies Inequality (III.8). That is to say, Inequality (III.109) holds. Plugging (III.109) into the first term, we obtain

$$\begin{aligned} |\varepsilon^{N-R} \langle \partial^{\alpha} (\varepsilon^{-N} \mathcal{L}^a(\varepsilon, v_{\varepsilon}^a)), \partial^{\alpha} v_{\varepsilon}^R \rangle| &\leq \frac{1}{2} \left(\varepsilon^{2(N-R)} \|\partial^{\alpha} (\varepsilon^{-N} \mathcal{L}^a(\varepsilon, v_{\varepsilon}^a))\|_{L^2}^2 + \|\partial^{\alpha} v_{\varepsilon}^R\|_{L^2}^2 \right), \\ &\leq \frac{1}{2} \left(C_{\mathcal{L}^a} \varepsilon^{2(N-R)} + \|v_{\varepsilon}^R\|_{H^{|\alpha|}}^2 \right). \end{aligned} \quad (\text{III.134})$$

◦ *The second term* is the contribution of the non linear part " $u \cdot \nabla u$ ". Select a positive constant c_1 . Further, we choose it so that the contribution of ∇v_{ε}^R in H^m -norm is small with respect to the dissipation. Then, we use the Gagliardo-Nirenberg inequality to deal with non-linear terms.

$$\begin{aligned} \varepsilon^{R+M-2} |\langle \partial^{\alpha} (v_{\varepsilon}^{1R} \partial_{\theta} v_{\varepsilon}^R + \varepsilon v_{\varepsilon}^{2R} \partial_y v_{\varepsilon}^R), \partial^{\alpha} v_{\varepsilon}^R \rangle| &\leq \frac{1}{2} \left(c_1 \varepsilon^{2(R+M-2)} \|\partial^{\alpha} (v_{\varepsilon}^{1R} \partial_{\theta} v_{\varepsilon}^R + v_{\varepsilon}^{2R} \partial_y v_{\varepsilon}^R)\|_{L^2}^2 + \frac{1}{c_1} \|\partial^{\alpha} v_{\varepsilon}^R\|_{L^2}^2 \right), \\ &\leq c_1 \varepsilon^{2(R+M-2)} \left(\|\partial^{\alpha} (v_{\varepsilon}^{1R} \partial_{\theta} v_{\varepsilon}^R)\|_{L^2}^2 + \|\partial^{\alpha} (v_{\varepsilon}^{2R} \partial_y v_{\varepsilon}^R)\|_{L^2}^2 \right) + \frac{1}{2c_1} \|\partial^{\alpha} v_{\varepsilon}^R\|_{L^2}^2. \end{aligned} \quad (\text{III.135})$$

We start by estimating the non-linear term $\partial^{\alpha} (v_{\varepsilon}^{1R} \partial_{\theta} v_{\varepsilon}^R)$. Applying Lemma III.19 together with (III.119), there exists a positive constant C_g which only depends on m such that for all

$\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\begin{aligned} \|\partial^\alpha (v_\varepsilon^{1R} \partial_\theta v_\varepsilon^R)\|_{L^2}^2 &\leq C_g^2 \left(\|v_\varepsilon^{1R}\|_{L^\infty} \|\partial_\theta v_\varepsilon^R\|_{\dot{H}^{|\alpha|}} + \|v_\varepsilon^{1R}\|_{\dot{H}^{|\alpha|}} \|\partial_\theta v_\varepsilon^R\|_{L^\infty} \right)^2, \\ &\leq 4 C_g^2 \left(\|v_\varepsilon^R\|_{\dot{H}^{|\alpha|+1}} + \|v_\varepsilon^R\|_{\dot{H}^{|\alpha|}} \right)^2 \leq 8 C_g^2 \left(\|v_\varepsilon^R\|_{\dot{H}^{|\alpha|+1}}^2 + \|v_\varepsilon^R\|_{\dot{H}^{|\alpha|}}^2 \right). \end{aligned}$$

In the same way we obtain for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\|\partial^\alpha (v_\varepsilon^{2R} \partial_y v_\varepsilon^R)\|_{L^2}^2 \leq 8 C_g^2 \left(\|v_\varepsilon^R\|_{\dot{H}^{|\alpha|+1}}^2 + \|v_\varepsilon^R\|_{\dot{H}^{|\alpha|}}^2 \right).$$

Finally, injecting the two previous majorations into Inequality (III.135) we get for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$

$$\begin{aligned} |\varepsilon^{R+M-2} \langle \partial^\alpha (v_\varepsilon^{1R} \partial_\theta v_\varepsilon^R + \varepsilon v_\varepsilon^{2R} \partial_y v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle| \\ \leq 16 c_1 C_g^2 \|v_\varepsilon^R\|_{\dot{H}^{|\alpha|+1}}^2 + (16 c_1 C_g^2 + \frac{1}{2c_1}) \|v_\varepsilon^R\|_{\dot{H}^{|\alpha|}}^2. \end{aligned} \quad (\text{III.136})$$

◦ *The last term.* It characterized the coupling between the velocity and the pressure through the term $q \nabla q$. At fixed ε it requires more regularity on the pressure. By an integration by parts we make this extra-derivative act on the velocity. Besides, the pressure is only expected to be controlled in the anisotropic Sobolev spaces $H_{(1,\varepsilon)}^m$. At a cost in a power of ε we can go back to the H^m -norm thanks to the equivalence of norms (III.106).

$$\begin{aligned} \left| -C 2^{-1} \varepsilon^{2\nu-M-R-2} \left\langle {}^t(\partial_\theta, \varepsilon \partial_y) \partial^\alpha (q_\varepsilon^a + \varepsilon^R q_\varepsilon^R)^2, \partial^\alpha v_\varepsilon^R \right\rangle \right| = \\ \left| C 2^{-1} \varepsilon^{2\nu-M-R-2} \left\langle \partial^\alpha (q_\varepsilon^a + \varepsilon^R q_\varepsilon^R)^2, \partial^\alpha {}^t(\partial_\theta, \varepsilon \partial_y) v_\varepsilon^R \right\rangle \right|. \end{aligned}$$

We select a positive constant c_2 , then

$$\begin{aligned} \left| -C 2^{-1} \varepsilon^{2\nu-M-R-2} \left\langle {}^t(\partial_\theta, \varepsilon \partial_y) \partial^\alpha (q_\varepsilon^a + \varepsilon^R q_\varepsilon^R)^2, \partial^\alpha v_\varepsilon^R \right\rangle \right| \\ \leq \frac{C^2 \varepsilon^{2(2\nu-R-M-2)}}{8c_2} \left\| \partial^\alpha (q_\varepsilon^a + \varepsilon^R q_\varepsilon^R)^2 \right\|_{L^2}^2 + \frac{c_2}{2} \left\| \partial^\alpha {}^t(\partial_\theta, \varepsilon \partial_y) v_\varepsilon^R \right\|_{L^2}^2. \end{aligned} \quad (\text{III.137})$$

We apply the Gagliardo-Nirenberg inequality together with the equivalence of norms (III.106)

$$\begin{aligned} \left\| \partial^\alpha (q_\varepsilon^a + \varepsilon^R q_\varepsilon^R)^2 \right\|_{L^2} &\leq 2C_g \|q_\varepsilon^a + \varepsilon^R q_\varepsilon^R\|_{L^\infty} \|q_\varepsilon^a + \varepsilon^R q_\varepsilon^R\|_{\dot{H}^{|\alpha|}}, \\ &\leq 2C_g \varepsilon^{-|\alpha|} \|q_\varepsilon^a + \varepsilon^R q_\varepsilon^R\|_{L^\infty} \|q_\varepsilon^a + \varepsilon^R q_\varepsilon^R\|_{\dot{H}_{(1,\varepsilon)}^{|\alpha|}}. \end{aligned}$$

With regards to the construction of the time T_ε^* , we have (III.108) and (III.119) *i.e.*

$$\forall \varepsilon \in]0, 1], \quad \forall t \in [0, T_\varepsilon^*], \quad \|q_\varepsilon^R(t, \cdot)\|_{\dot{H}_{(1,\varepsilon)}^{|\alpha|}} \leq 2, \quad \|q_\varepsilon^a(t, \cdot)\|_{\dot{H}_{(1,\varepsilon)}^{|\alpha|}} \leq C_a,$$

and it results in:

$$\forall \varepsilon \in]0, 1], \quad \forall t \in [0, T_\varepsilon^*], \quad \|\sqrt{\varepsilon} q_\varepsilon^R(t, \cdot)\|_{L^\infty(\mathbb{T} \times \mathbb{R})} \leq 2, \quad \|\sqrt{\varepsilon} q_\varepsilon^a(t, \cdot)\|_{L^\infty(\mathbb{T} \times \mathbb{R})} \leq C_a.$$

We obtain

$$\left\| \partial^\alpha (q_\varepsilon^a + \varepsilon^R q_\varepsilon^R)^2 \right\|_{L^2} \leq 2 C_g \varepsilon^{-1/2-|\alpha|} (C_a + 2)^2. \quad (\text{III.138})$$

Finally plugging the above estimate (III.138) in (III.137), we deduce:

$$\begin{aligned} & \left| -C 2^{-1} \varepsilon^{2\nu-M-R-2} \left\langle {}^t(\partial_\theta, \varepsilon \partial_y) \partial^\alpha (q_\varepsilon^a + \varepsilon^R q_\varepsilon^R)^2, \partial^\alpha v_\varepsilon^R \right\rangle \right| \\ & \leq \frac{C^2 C_g^2 \varepsilon^{2(2\nu-R-M-5/2-|\alpha|)}}{2c_2} (C_a + 2)^4 + \frac{c_2}{2} \|v_\varepsilon^R\|_{\dot{H}^{|\alpha|+1}}^2. \end{aligned} \quad (\text{III.139})$$

◦ *To finish*, we put estimates (III.134), (III.136) and (III.139) together. Let c_1 and c_2 be two positive constants for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\begin{aligned} |\langle \partial^\alpha S_\varepsilon^{R,N}, \partial^\alpha v_\varepsilon^R \rangle| & \leq \left(\frac{c_2}{2} + 16 c_1 C_g^2 \right) \|v_\varepsilon^R\|_{\dot{H}^{|\alpha|+1}}^2 + \left(\frac{1}{2} + 16 c_1 C_g^2 + \frac{1}{2c_1} \right) \|v_\varepsilon^R\|_{\dot{H}^{|\alpha|}}^2 \\ & \quad + \left(C_{\mathcal{L}^a} \varepsilon^{2(N-R)} + \frac{C^2 C_g^2}{2c_2} (C_a + 2)^4 \varepsilon^{2(2\nu-R-M-5/2-|\alpha|)} \right), \\ & \leq \left(\frac{c_2}{2} + 16 c_1 C_g^2 \right) \|v_\varepsilon^R\|_{\dot{H}^{|\alpha|+1}}^2 + \left(\frac{1}{2} + 16 c_1 C_g^2 + \frac{1}{2c_1} \right) \|v_\varepsilon^R\|_{\dot{H}^{|\alpha|}}^2 \\ & \quad + \left(C_{\mathcal{L}^a} + \frac{C^2 C_g^2}{2c_2} (C_a + 2)^4 \right) \varepsilon^{w_m}. \end{aligned}$$

Select C_S a positive constant, there exist c_1 and c_2 two positive constants such that $C_S := \frac{c_2}{2} + 16 c_1 C_g^2$. Then, it requires $C_S^1 = \frac{1}{2} + 16 c_1 C_g^2 + \frac{1}{2c_1}$ and $C_p = C_{\mathcal{L}^a} + \frac{C C_g^2}{2c_2} (C_a + 2)^4$. ■

•**Proof of Lemma III.22**• We now move to the proof of Lemma III.22. We perform L^2 -estimates for Equation (III.126). We prove the result by studying the competition between the dissipation and the singular terms (with respect to ε or the number of derivatives). We decompose the proof between the case $J = 0$ and the case $J \in \{1, \dots, m+3\}$. The proof is essentially the same in both cases. In concern of accuracy, we produce both.

◦ *The case $J = 0$* . We multiply Equation (III.126) by v_ε^R and integrate. Applying Lemma III.25, there exists $\varepsilon_d > 0$ such that for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$,

$$\begin{aligned} \frac{1}{2} \partial_t \|v_\varepsilon^R\|_{L^2}^2 - \langle Q_{\varepsilon, \lambda_\varepsilon} v_\varepsilon^R, v_\varepsilon^R \rangle & = \langle S_\varepsilon^{R,N}, v_\varepsilon^R \rangle - \langle \mathcal{A} v_\varepsilon^R, v_\varepsilon^R \rangle - \langle \mathcal{B} v_\varepsilon^R, v_\varepsilon^R \rangle - \langle \mathcal{C} v_\varepsilon^R, v_\varepsilon^R \rangle \\ & \geq \frac{1}{2} \|\partial_t v_\varepsilon^R\|_{L^2}^2 + c_0 \Phi_\varepsilon(\nabla, v_\varepsilon^R). \end{aligned}$$

-We start by absorbing the contribution of operator \mathcal{C} . Select c_1 a positive constant to be determined further. According to Corollary III.26 we obtain for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$,

$$\begin{aligned} \frac{1}{2} \partial_t \|v_\varepsilon^R\|_{L^2}^2 + \left(c_0 - \frac{c_1 \|h\|_{L^\infty}}{2} \right) \Phi_\varepsilon(\nabla, v_\varepsilon^R) \\ \leq |\langle S_\varepsilon^{R,N}, v_\varepsilon^R \rangle| + |\langle \mathcal{A} v_\varepsilon^R, v_\varepsilon^R \rangle| + |\langle \mathcal{B} v_\varepsilon^R, v_\varepsilon^R \rangle| + \frac{\|h\|_{L^\infty}}{2c_1} \|v_\varepsilon^R\|_{L^2}^2. \end{aligned}$$

We choose c_1 so that $\tilde{c}_0 := c_0 - \frac{c_1 \|h\|_{L^\infty}}{2} > 0$. Then for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$

$$\frac{1}{2} \partial_t \|v_\varepsilon^R\|_{L^2}^2 + \tilde{c}_0 \Phi_\varepsilon(\nabla, v_\varepsilon^R) \leq |\langle S_\varepsilon^{R,N}, v_\varepsilon^R \rangle| + |\langle \mathcal{A}v_\varepsilon^R, v_\varepsilon^R \rangle| + |\langle \mathcal{B}v_\varepsilon^R, v_\varepsilon^R \rangle| + \frac{\|h\|_{L^\infty}}{2c_1} \|v_\varepsilon^R\|_{L^2}^2.$$

-There remains to deal with the degenerate terms contained in $S_\varepsilon^{R,N}$. Select C_S a positive constant to be chosen later. According to Lemma III.29, there exist C_S^1 and C_p such that for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$,

$$\begin{aligned} \frac{1}{2} \partial_t \|v_\varepsilon^R(t, \cdot)\|_{L^2}^2 + \tilde{c}_0 \Phi_\varepsilon(\nabla, v_\varepsilon^R) - C_S \|v_\varepsilon^R(t, \cdot)\|_{H^1}^2 \leq \\ |\langle \mathcal{A}v_\varepsilon^R, v_\varepsilon^R \rangle| + |\langle \mathcal{B}v_\varepsilon^R, v_\varepsilon^R \rangle| + \left(C_S^1 + \frac{\|h\|_{L^\infty}}{2c_1} \right) \|v_\varepsilon^R(t, \cdot)\|_{L^2}^2 + C_p \varepsilon^{2w_m}. \end{aligned}$$

We choose C_S such that $\tilde{c}_1 := \tilde{c}_0 - C_S > 0$. In that case for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\tilde{c}_1 \Phi_\varepsilon(\nabla, v_\varepsilon^R) \leq \tilde{c}_0 \Phi_\varepsilon(\nabla, v_\varepsilon^R) - C_S \|v_\varepsilon^R(t, \cdot)\|_{H^1}^2.$$

-According to Lemma III.23 and Lemma III.24, there exist $C_{\mathcal{A}}^1$ and $C_{\mathcal{B}}^1$ two positive constants such that for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$,

$$|\langle \mathcal{A}v_\varepsilon^R, v_\varepsilon^R \rangle| \leq C_{\mathcal{A}}^1 \|v_\varepsilon^R(t, \cdot)\|_{L^2}^2, \quad |\langle \mathcal{B}v_\varepsilon^R, v_\varepsilon^R \rangle| \leq C_{\mathcal{B}}^1 \|v_\varepsilon^R(t, \cdot)\|_{L^2}^2.$$

-Finally, putting everything together we get that for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$,

$$\frac{1}{2} \partial_t \|v_\varepsilon^R(t, \cdot)\|_{L^2}^2 + \tilde{c}_1 \Phi_\varepsilon(\nabla, v_\varepsilon^R) \leq \left(C_S^1 + C_{\mathcal{A}}^1 + C_{\mathcal{B}}^1 + \frac{\|h\|_{L^\infty}}{2c_1} \right) \|v_\varepsilon^R(t, \cdot)\|_{L^2}^2 + C_p \varepsilon^{2w_m}.$$

□

◦ *The case $J \in \{1, \dots, m+3\}$.* Choose a multi-index $\alpha \in \mathbb{N}^2$ such that $1 \leq |\alpha| \leq m+3$. The beginning of the proof is the same as the case $J = 0$. Notwithstanding, we need to absorb the H^{J+1} -norm of the velocity thanks to Φ_ε . That is no longer possible. Indeed, $\Phi_\varepsilon(\nabla, \partial^\alpha v_\varepsilon^R)$ only contains the derivatives $\partial^\alpha \partial_\theta$ and $\partial^\alpha \partial_y$. All other derivatives of order $J+1$ are missing in order to absorb the homogeneous semi-norm $\|\cdot\|_{H^{J+1}}^\circ$. We make those missing derivatives appear considering the new quadratic form:

$$\sum_{|\alpha|=J+1} \Phi_\varepsilon(\nabla, \partial^\alpha v_\varepsilon^R). \quad (\text{III.140})$$

-First we derivate α times Equation (III.126). Then we multiply by $\partial^\alpha v_\varepsilon^R$ and integrate,

$$\begin{aligned} \frac{1}{2} \partial_t \|\partial^\alpha v_\varepsilon^R\|_{L^2}^2 - \langle Q_{\varepsilon, \lambda_\varepsilon} \partial^\alpha v_\varepsilon^R, \partial^\alpha v_\varepsilon^R \rangle &= \langle \partial^\alpha S_\varepsilon^{R,N}, \partial^\alpha v_\varepsilon^R \rangle \\ &\quad - \langle \partial^\alpha (\mathcal{A}v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle - \langle \partial^\alpha (\mathcal{B}v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle \\ &\quad + \langle ([\partial^\alpha, \mathcal{C}] v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle + \langle ([\partial^\alpha, \mathcal{H}] v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle - \langle \mathcal{H} \partial^\alpha v_\varepsilon^R, \partial^\alpha v_\varepsilon^R \rangle - \langle \mathcal{C} \partial^\alpha v_\varepsilon^R, \partial^\alpha v_\varepsilon^R \rangle. \end{aligned}$$

Applying Lemma III.25, there exists $\varepsilon_d > 0$ such that for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$

$$c_0 \Phi_\varepsilon (\nabla, \partial^\alpha v_\varepsilon^R) \leq - \langle Q_{\varepsilon, \lambda_\varepsilon} \partial^\alpha v_\varepsilon^R, \partial^\alpha v_\varepsilon^R \rangle.$$

-Select c_1 a positive constant to be chosen (small) later. Applying Lemma III.26, the operator \mathcal{C} can be absorbed thanks to Φ_ε , for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\begin{aligned} \frac{1}{2} \partial_t \|\partial^\alpha v_\varepsilon^R\|_{L^2}^2 + \left(c_0 - \frac{c_1 \|h\|_{L^\infty}}{2} \right) \Phi_\varepsilon (\nabla, \partial^\alpha v_\varepsilon^R) &\leq |\langle \partial^\alpha S_\varepsilon^{R,N}, \partial^\alpha v_\varepsilon^R \rangle| + \frac{\|h\|_{L^\infty}}{2c_1} \|\partial^\alpha v_\varepsilon^R\|_{L^2}^2 \\ &+ |\langle \partial^\alpha (\mathcal{A} v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle| + |\langle \partial^\alpha (\mathcal{B} v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle| \\ &+ |\langle ([\partial^\alpha, \mathcal{C}] v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle| + |\langle ([\partial^\alpha, \mathcal{H}] v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle|. \end{aligned}$$

We choose c_1 such that $\tilde{c}_0 := c_0 - \frac{c_1 \|h\|_{L^\infty}}{2} > 0$ so the above inequality becomes for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\begin{aligned} \frac{1}{2} \partial_t \|\partial^\alpha v_\varepsilon^R\|_{L^2}^2 + \tilde{c}_0 \Phi_\varepsilon (\nabla, \partial^\alpha v_\varepsilon^R) &\leq |\langle \partial^\alpha S_\varepsilon^{R,N}, \partial^\alpha v_\varepsilon^R \rangle| + \frac{\|h\|_{L^\infty}}{2c_1} \|\partial^\alpha v_\varepsilon^R\|_{L^2}^2 \\ &+ |\langle \partial^\alpha (\mathcal{A} v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle| + |\langle \partial^\alpha (\mathcal{B} v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle| \\ &+ |\langle ([\partial^\alpha, \mathcal{C}] v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle| + |\langle ([\partial^\alpha, \mathcal{H}] v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle|. \quad (\text{III.141}) \end{aligned}$$

-The second step is to deal with $S_\varepsilon^{R,N}$. It is controlled thanks to the H^{J+1} -norm of v_ε^R . We make it small enough so that the dissipation absorbs it. We select C_S a positive constant and from Lemma III.29, there exist two positive constants C_S^1 and C_p such that

$$|\langle \partial^\alpha S_\varepsilon^{R,N}, \partial^\alpha v_\varepsilon^R \rangle| (t, \cdot) \leq C_S \|v_\varepsilon^R(t, \cdot)\|_{H^{J+1}(\mathbb{T} \times \mathbb{R})}^2 + C_S^1 \|v_\varepsilon^R(t, \cdot)\|_{H^J(\mathbb{T} \times \mathbb{R})}^2 + C_p \varepsilon^{2w_m}.$$

It can be injected in Equation (III.141) and for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\begin{aligned} \frac{1}{2} \partial_t \|\partial^\alpha v_\varepsilon^R(t, \cdot)\|_{L^2}^2 + \tilde{c}_0 \Phi_\varepsilon (\nabla, \partial^\alpha v_\varepsilon^R) - C_S \|v_\varepsilon^R(t, \cdot)\|_{H^{J+1}(\mathbb{T} \times \mathbb{R})}^2 \\ \leq \frac{\|h\|_{L^\infty}}{2c_1} \|\partial^\alpha v_\varepsilon^R(t, \cdot)\|_{L^2}^2 + C_S^1 \|v_\varepsilon^R(t, \cdot)\|_{H^J(\mathbb{T} \times \mathbb{R})}^2 + C_p \varepsilon^{2w_m} \\ + |\langle \partial^\alpha (\mathcal{A} v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle| + |\langle \partial^\alpha (\mathcal{B} v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle| \\ + |\langle ([\partial^\alpha, \mathcal{C}] v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle| + |\langle ([\partial^\alpha, \mathcal{H}] v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle|. \end{aligned}$$

We sum over all $\alpha \in \mathbb{N}^2$ of length J so that the quadratic form (III.140) becomes apparent. For all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$, we have

$$\begin{aligned} \frac{1}{2} \partial_t \|v_\varepsilon^R(t, \cdot)\|_{H^J}^2 + \sum_{|\alpha|=J} \tilde{c}_0 \Phi_\varepsilon (\nabla, \partial^\alpha v_\varepsilon^R) - (J+1) C_S \|v_\varepsilon^R(t, \cdot)\|_{H^{J+1}(\mathbb{T} \times \mathbb{R})}^2 \\ \leq \frac{\|h\|_{L^\infty}}{2c_1} \|v_\varepsilon^R(t, \cdot)\|_{H^J}^2 + (J+1) C_S^1 \|v_\varepsilon^R(t, \cdot)\|_{H^J(\mathbb{T} \times \mathbb{R})}^2 + (J+1) C_p \varepsilon^{2w_m} \\ + \sum_{|\alpha|=J} |\langle \partial^\alpha (\mathcal{A} v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle| + \sum_{|\alpha|=J} |\langle \partial^\alpha (\mathcal{B} v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle| \\ + \sum_{|\alpha|=J} |\langle ([\partial^\alpha, \mathcal{C}] v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle| + \sum_{|\alpha|=J} |\langle ([\partial^\alpha, \mathcal{H}] v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle|. \end{aligned}$$

Let C_S be a positive constant such that $\tilde{c}_1 := \tilde{c}_0 - (m+4)C_S > 0$. In that case:

$$\sum_{|\alpha|=J} \tilde{c}_0 \Phi_\varepsilon(\nabla, \partial^\alpha v_\varepsilon^R) - (J+1)C_S \|v_\varepsilon^R(t, \cdot)\|_{H^{J+1}(\mathbb{T} \times \mathbb{R})}^2 \geq \tilde{c}_1 \sum_{|\alpha|=J} \Phi_\varepsilon(\nabla, \partial^\alpha v_\varepsilon^R).$$

Finally we obtain for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\begin{aligned} & \frac{1}{2} \partial_t \|v_\varepsilon^R(t, \cdot)\|_{H^J}^2 + \tilde{c}_1 \sum_{|\alpha|=J} \Phi_\varepsilon(\nabla, \partial^\alpha v_\varepsilon^R) \\ & \leq \frac{\|h\|_{L^\infty}}{2c_1} \|v_\varepsilon^R(t, \cdot)\|_{H^J}^2 + (J+1)C_S^1 \|v_\varepsilon^R(t, \cdot)\|_{H^J(\mathbb{T} \times \mathbb{R})}^2 + (J+1)C_p \varepsilon^{2w_m} \\ & \quad + \sum_{|\alpha|=J} |\langle \partial^\alpha (\mathcal{A} v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle| + \sum_{|\alpha|=J} |\langle \partial^\alpha (\mathcal{B} v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle| \\ & \quad + \sum_{|\alpha|=J} |\langle [\partial^\alpha, \mathcal{C}] v_\varepsilon^R, \partial^\alpha v_\varepsilon^R \rangle| + \sum_{|\alpha|=J} |\langle [\partial^\alpha, \mathcal{H}] v_\varepsilon^R, \partial^\alpha v_\varepsilon^R \rangle|. \end{aligned}$$

-There remains to majorate the contribution of \mathcal{A} , \mathcal{B} and the commutators $[\partial^\alpha, \mathcal{H}]$ and $[\partial^\alpha, \mathcal{C}]$. They no longer bring terms with derivatives of order $J+1$. From Lemma III.23, Lemma III.24, Lemma III.27 and Lemma III.28 there exist constants $C_{\mathcal{A}}^1$, $C_{\mathcal{A}}^2$ and $C_{\mathcal{B}}^1$, $C_{\mathcal{B}}^2$, and $C_{\mathcal{H}}^1$, $C_{\mathcal{H}}^2$, $C_{\mathcal{H}}^3$, and $C_{\mathcal{C}}^1$, $C_{\mathcal{C}}^2$, $C_{\mathcal{C}}^3$ all positive such that for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\begin{aligned} |\langle \partial^\alpha (\mathcal{A} v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle| & \leq C_{\mathcal{A}}^1 \|v_\varepsilon^R\|_{H^J}^2 + C_{\mathcal{A}}^2 \|v_\varepsilon^R\|_{H^{J-1}}^2, \\ |\langle \partial^\alpha (\mathcal{B} v_\varepsilon^R), \partial^\alpha v_\varepsilon^R \rangle| & \leq C_{\mathcal{B}}^1 \|v_\varepsilon^R\|_{H^J}^2 + C_{\mathcal{B}}^2 \|v_\varepsilon^R\|_{H^{J-1}}^2, \\ |\langle [\partial^\alpha, \mathcal{H}] v_\varepsilon^R, \partial^\alpha v_\varepsilon^R \rangle| & \leq C_{\mathcal{H}}^1 \|v_\varepsilon^R\|_{H^J}^2 + C_{\mathcal{H}}^2 \|v_\varepsilon^R\|_{H^{J-1}}^2 + C_{\mathcal{H}}^3 \left\| \varepsilon^{-(1-\delta_{\alpha_1, 0})} \partial^\alpha v_\varepsilon^R \right\|_{L^2(\mathbb{T} \times \mathbb{R})}^2, \\ |\langle [\partial^\alpha, \mathcal{C}] v_\varepsilon^R, \partial^\alpha v_\varepsilon^R \rangle| & \leq C_{\mathcal{C}}^1 \|v_\varepsilon^R\|_{H^J}^2 + C_{\mathcal{C}}^2 \|v_\varepsilon^R\|_{H^{J-1}}^2 + C_{\mathcal{C}}^3 \left\| \varepsilon^{-(1-\delta_{\alpha_1, 0})} \partial^\alpha v_\varepsilon^R \right\|_{L^2(\mathbb{T} \times \mathbb{R})}^2. \end{aligned}$$

Putting everything together, we finally deduce that for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\begin{aligned} \frac{1}{2} \partial_t \|v_\varepsilon^R\|_{H^J}^2 + \tilde{c}_1 \sum_{|\alpha|=J} \Phi_\varepsilon(\nabla, \partial^\alpha v_\varepsilon^R) & \leq \left(\frac{\|h\|_{L^\infty}}{2c_1} + (J+1)(C_S^1 + C_{\mathcal{A}}^1 + C_{\mathcal{B}}^1 + C_{\mathcal{C}}^1 + C_{\mathcal{H}}^1) \right) \|v_\varepsilon^R\|_{H^J}^2 \\ & \quad + (J+1)(C_{\mathcal{A}}^2 + C_{\mathcal{B}}^2 + C_{\mathcal{C}}^2 + C_{\mathcal{H}}^2) \|v_\varepsilon^R\|_{H^{J-1}}^2 \\ & \quad + (C_{\mathcal{H}}^3 + C_{\mathcal{C}}^3) \sum_{|\alpha|=J} \left\| \varepsilon^{-(1-\delta_{\alpha_1, 0})} \partial^\alpha v_\varepsilon^R \right\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 + (J+1)C_p \varepsilon^{2w_m}. \end{aligned}$$

■

3.1.2 Control over the velocity

This subsection is dedicated to the proof of Proposition III.21. We prove by induction that property $\mathcal{P}(J)$, defined at the level of (III.124), is satisfied for $J \in \{0, \dots, m+3\}$.

◦ *Condition $\mathcal{P}(0)$ is true.* We apply Lemma III.22 in the case $J = 0$. There exist ε_d and c_1 two positive constants, there exist C_0^1 and C_p two positive constants such that for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\frac{1}{2} \partial_t \|v_\varepsilon^R(t, \cdot)\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 + c_1 \Phi_\varepsilon(\nabla, v_\varepsilon^R)(t) \leq C_0^2 \|v_\varepsilon^R(t, \cdot)\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 + C_p \varepsilon^{2w_m}.$$

-The quadratic form Φ_ε is positive. First neglecting its contribution we get that for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\frac{1}{2} \partial_t \|v_\varepsilon^R(t, \cdot)\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 \leq C_0^2 \|v_\varepsilon^R(t, \cdot)\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 + C_p \varepsilon^{2w_m}.$$

We apply the Gronwall lemma to obtain:

$$\forall \varepsilon \in]0, \varepsilon_d], \forall t \in [0, T_\varepsilon^*], \quad \|v_\varepsilon^R(t, \cdot)\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 \leq \int_0^t e^{(t-s)C_0^2} C_p \varepsilon^{2w_m} ds.$$

Finally, there exist K_0^1 and K_0^2 such that for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\|v_\varepsilon^R(t, \cdot)\|_{L^2}^2 \leq \varepsilon^{2w_m} K_0^1 \left(e^{K_0^2 t} - 1 \right). \quad (\text{III.142})$$

- To obtain the regularity $L^2([0, T_\varepsilon^*], H^1)$ (over the velocity), we go back to Equation (III.125). Integrating with respect to t and since $\|v_\varepsilon^R(t, \cdot)\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 \geq 0$ we get for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$c_1 \int_0^t \Phi_\varepsilon(\nabla, v_\varepsilon^R)(s) ds \leq C_0^2 \int_0^t \|v_\varepsilon^R(s, \cdot)\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 ds + t C_p \varepsilon^{2w_m}.$$

From Equation (III.142), there exists $K_0^3 > 0$ such that

$$\forall \varepsilon \in]0, \varepsilon_d], \quad \forall t \in [0, T_\varepsilon^*], \quad \int_0^t \Phi_\varepsilon(\nabla, v_\varepsilon^R)(s) ds \leq \varepsilon^{2w_m} K_0^3 t.$$

□

◦ *We assume that $\mathcal{P}(J)$ is true for $J \in \{0, \dots, m+2\}$.* We start following the same step as the previous case $J = 0$. We apply Lemma III.22 in the case $J+1$ (≥ 1). There exist ε_d and c_1 two positive constants, there exist C_{J+1}^1 , C_{J+1}^2 , C_{J+1}^3 and C_p four positive constants such that for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\begin{aligned} & \frac{1}{2} \partial_t \|v_\varepsilon^R(t, \cdot)\|_{H^{J+1}(\mathbb{T} \times \mathbb{R})}^2 + c_1 \sum_{|\alpha|=J+1} \Phi_\varepsilon(\nabla, \partial^\alpha v_\varepsilon^R)(t) \\ & \leq C_{J+1}^1 \|v_\varepsilon^R(t, \cdot)\|_{H^J(\mathbb{T} \times \mathbb{R})}^2 + C_{J+1}^2 \|v_\varepsilon^R(t, \cdot)\|_{H^{J+1}(\mathbb{T} \times \mathbb{R})}^2 \\ & + C_{J+1}^3 \sum_{|\alpha|=J+1} \left\| \varepsilon^{-(1-\delta_{\alpha 1, 0})} \partial^\alpha v_\varepsilon^R(t, \cdot) \right\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 + (J+2) C_p \varepsilon^{2w_m}. \quad (\text{III.143}) \end{aligned}$$

The main difference with the previous case $J = 0$ is the appearance of the degerate term

$$\sum_{|\alpha|=J+1} \left\| \varepsilon^{-(1-\delta_{\alpha_1,0})} \partial^\alpha v_\varepsilon^R(t, \cdot) \right\|_{L^2(\mathbb{T} \times \mathbb{R})}^2.$$

Yet according to the assumption of induction $\mathcal{P}(J)$ we can interpret it as a source term. We precise it further.

-We first look for the H^{J+1} -estimate. We neglect $\sum_{|\alpha|=J+1} \Phi_\varepsilon(\nabla, \partial^\alpha v_\varepsilon^R)$ since it is positive as a sum of positive quadratic forms. Hence, for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\begin{aligned} \frac{1}{2} \partial_t \|v_\varepsilon^R(t, \cdot)\|_{H^{J+1}(\mathbb{T} \times \mathbb{R})}^2 &\leq C_{J+1}^1 \|v_\varepsilon^R(t, \cdot)\|_{H^J(\mathbb{T} \times \mathbb{R})}^2 + C_{J+1}^2 \|v_\varepsilon^R(t, \cdot)\|_{H^{J+1}(\mathbb{T} \times \mathbb{R})}^2 \\ &\quad + C_{J+1}^3 \sum_{|\alpha|=J+1} \left\| \varepsilon^{-(1-\delta_{\alpha_1,0})} \partial^\alpha v_\varepsilon^R(t, \cdot) \right\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 + (J+2) C_p \varepsilon^{2w_m}. \end{aligned}$$

We set for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\begin{aligned} S_{J+1}(t) &:= C_{J+1}^1 \|v_\varepsilon^R(t, \cdot)\|_{H^J(\mathbb{T} \times \mathbb{R})}^2 \\ &\quad + C_{J+1}^3 \sum_{|\alpha|=J+1} \left\| \varepsilon^{-(1-\delta_{\alpha_1,0})} \partial^\alpha v_\varepsilon^R(t, \cdot) \right\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 + (J+2) C_p \varepsilon^{2w_m}. \end{aligned}$$

According to Inequality (III.122), *i.e.* the first part of assumption $\mathcal{P}(J)$, the function $t \mapsto \|v_\varepsilon^R(t, \cdot)\|_{H^J}^2$ lies in $L^1([0, T_\varepsilon^*])$. In addition there exists C_m^{11} a positive constant such that for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\|v_\varepsilon^R(t, \cdot)\|_{H^J}^2 \leq C_m^{11} \varepsilon^{2w_m}.$$

Similarly according to (III.123), the second part of assumption $\mathcal{P}(J)$, the function

$$t \mapsto \sum_{|\alpha|=J+1} \left\| \varepsilon^{-(1-\delta_{\alpha_1,0})} \partial^\alpha v_\varepsilon^R(t, \cdot) \right\|_{L^2(\mathbb{T} \times \mathbb{R})}^2$$

is in $L^1([0, T_\varepsilon^*])$. Moreover there exists C_m^{12} a positive constant such that for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\sum_{|\alpha|=J+1} \left\| \varepsilon^{-(1-\delta_{\alpha_1,0})} \partial^\alpha v_\varepsilon^R(t, \cdot) \right\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 \leq C_m^{12} \varepsilon^{2w_m}.$$

Thus, the function $t \mapsto S_{J+1}(t)$ is in $L^1([0, T_\varepsilon^*])$. There exists $M_J^1 := C_m^{11} + C_m^{12} + C_p$ a positive constant such that for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\begin{aligned} C_{J+1}^1 \|v_\varepsilon^R(t, \cdot)\|_{H^J(\mathbb{T} \times \mathbb{R})}^2 \\ + C_{J+1}^3 \sum_{|\alpha|=J+1} \left\| \varepsilon^{-(1-\delta_{\alpha_1,0})} \partial^\alpha v_\varepsilon^R(t, \cdot) \right\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 + C_p \varepsilon^{2w_m} \leq M_J^1 \varepsilon^{2w_m}. \end{aligned}$$

We can apply the Gronwall's lemma, for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$,

$$\begin{aligned} \|v_\varepsilon^R(t, \cdot)\|_{H^{J+1}(\mathbb{T} \times \mathbb{R})}^2 &\leq \int_0^t e^{2C_{J+1}^2(t-s)} \left(C_{J+1}^1 \|v_\varepsilon^R(s, \cdot)\|_{H^J(\mathbb{T} \times \mathbb{R})}^2 \right. \\ &\quad \left. + C_{J+1}^3 \sum_{|\alpha|=J+1} \left\| \varepsilon^{-(1-\delta_{\alpha_1,0})} \partial^\alpha v_\varepsilon^R(s, \cdot) \right\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 + (J+2) C_p \varepsilon^{2w_m} \right) (s) ds \\ &\leq M_J^1 \varepsilon^{2w_m} \int_0^t e^{2C_{J+1}^2(t-s)} ds \leq \frac{M_J^1}{2C_{J+1}^2} \varepsilon^{2w_m} \left(e^{2C_{J+1}^2 t} - 1 \right). \end{aligned}$$

Finally there exists $K_{J+1}^1 := M_J^1/(2C_{J+1}^2)$ and $K_{J+1}^2 := 2C_{J+1}^2$ two positive constants such that for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$,

$$\|v_\varepsilon^R(t, \cdot)\|_{H^{J+1}(\mathbb{T} \times \mathbb{R})}^2 \leq \varepsilon^{2w_m} K_{J+1}^1 \left(e^{K_{J+1}^2 t} - 1 \right).$$

-There remains to get the estimation $L^2([0, T_\varepsilon^*], H^{J+2})$. We go back to Equation (III.143) and integrate it with respect to the time t ,

$$\int_0^t \sum_{|\alpha|=J+1} \Phi_\varepsilon(\nabla, \partial^\alpha v_\varepsilon^R)(s) ds \leq C_{J+1}^2 \int_0^t \|v_\varepsilon^R(s, \cdot)\|_{H^{J+1}}^2 ds + \int_0^t S_{J+1}(s) ds.$$

According to the previous discussion, the function $t \mapsto \|v_\varepsilon^R(t, \cdot)\|_{H^{J+1}}^2$ is in $L^1([0, T_\varepsilon^*])$. Furthermore from the first part of the proof, there exists C_{J+1}^{31} such that for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\|v_\varepsilon^R(t, \cdot)\|_{H^{J+1}}^2 \leq C_{J+1}^{31} \varepsilon^{2w_m} t \leq C_{J+1}^{31} \varepsilon^{2w_m}.$$

We already prove that $S_{J+1} \in L^1([0, T_\varepsilon^*])$ and $S_{J+1}(t) \leq M_J^1 \varepsilon^{2w_m}$ on $[0, T_\varepsilon^*]$. Finally there exists $K_{J+1}^3 > 0$ such that for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\int_0^t \sum_{|\alpha|=J+1} \Phi_\varepsilon(\nabla, \partial^\alpha v_\varepsilon^R)(s) ds \leq \varepsilon^{2w_s} K_{J+1}^3 t.$$

This ends the proof. ■

Remark. In the above discussion, we only use the assumption $M \geq 2$ whereas we assume $M \geq 7/2$. It becomes crucial when we estimate the pressure.

3.2 Control over the pressure

In Subsection 3.2 we prove estimate (III.121) (for the pressure). The main difference with the velocity case is that Equation (III.117a) is a pure hyperbolic type equation. We do not expect an *absorption phenomena* neither a *regularization phenomena* induced by the dissipation. It has two direct consequences.

- At fixed ε , Equation (III.117a) contains derivatives of order one of the pressure $\partial_y q$ or $\partial_\theta q$. To deal with those singular terms (with respect to the regularity) we integrate by parts (again) when necessary to pass the extra-derivatives on the velocity. If we perform L^2 estimates, it requires H^1 estimates on the velocity. The regularization of the viscosity (III.123) plays a crucial role in this process.
- We can not control the singular transport $\varepsilon^{-1} h \partial_y$ in classical Sobolev spaces. We introduce the anisotropic Sobolev spaces $H_{(1,\varepsilon)}^m$. The choice of those spaces for the pressure produce an other issue. Consider $\{f_\varepsilon\}_\varepsilon$ a family of function bounded in $H_{(1,\varepsilon)}^m$. Then, the family $\{\partial_\theta f_\varepsilon\}_\varepsilon$ is singular in L^2 . Thus, we can not control terms such as $\varepsilon^{M-2} v_\varepsilon^{1a} \partial_\theta q_\varepsilon^R$ or $\varepsilon^{M-2} v_\varepsilon^{1R} \partial_\theta q_\varepsilon^R$ in Equation (III.117a). We absorb those new singularities (due to the choice of spaces) thanks to M assuming $M \geq 3$ (at least).

Here we prove the following control in time over the pressure.

Proposition III.30. *Let q_ε^R be a solution of Equation (III.117a) on $[0, T_\varepsilon^*]$. There exist a positive constant ε_{crit} and a positive constant K_m^1 (independent of ε) such that*

$$\forall \varepsilon \in]0, \varepsilon_{crit}], \quad \forall t \in [0, T_\varepsilon^*], \quad \|q_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^{m+3}(\mathbb{T} \times \mathbb{R})}^2 \leq \varepsilon^{2w_m} K_m^1 t.$$

Again, we prove the result by induction on the size m setting $\mathcal{Q}(m)$:

$$\mathcal{Q}(m) : \text{ " The Proposition III.30 is satisfied up to the integer } m \text{ " .} \quad (\text{III.144})$$

To go from m to $m+1$, the proof is once more based on an energy method for Equation (III.117a) in anisotropic Sobolev spaces $H_{(1,\varepsilon)}^m$.

Lemma III.31. *There exists a positive constant $\varepsilon_d > 0$ such that for any $J \in \{0, \dots, m+3\}$, the following assertions hold.*

i) *If $J = 0$, there exist two positive constants C_0^1 and C_0^3 such that for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:*

$$\begin{aligned} \frac{1}{2} \partial_t \|q_\varepsilon^R(t, \cdot)\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 &\leq C_0^1 \left(1 + \|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}(\mathbb{T} \times \mathbb{R})}^2 \right) \|q_\varepsilon^R(t, \cdot)\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 \\ &\quad + C_0^3 \left(\|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}(\mathbb{T} \times \mathbb{R})}^2 + \varepsilon^{2(N-R)} \right). \end{aligned} \quad (\text{III.145})$$

ii) *If $J \geq 1$, there exist three positive constants C_J^1 , C_J^2 and C_J^3 such that for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:*

$$\begin{aligned} \frac{1}{2} \partial_t \|q_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^J(\mathbb{T} \times \mathbb{R})}^2 &\leq C_J^1 \left(1 + \|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}(\mathbb{T} \times \mathbb{R})}^2 \right) \|q_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^J(\mathbb{T} \times \mathbb{R})}^2 \\ &\quad + C_J^2 \left(1 + \|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}(\mathbb{T} \times \mathbb{R})}^2 \right) \|q_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^{J-1}(\mathbb{T} \times \mathbb{R})}^2 \\ &\quad + C_J^3 \left(\|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}(\mathbb{T} \times \mathbb{R})}^2 + \varepsilon^{2(N-R)} \right). \end{aligned} \quad (\text{III.146})$$

As mentionned above, as we perform *a priori* H^m -estimates it requires the velocity to be in $L^\infty([0, T_\varepsilon], H^{m+4}(\mathbb{T} \times \mathbb{R}))$. This is not the case. However it has more meaning, if we integrate Inequalities (III.145) and (III.146) with respect to t . It just requires the velocity to lie in $L^2([0, T_\varepsilon^*], H^{m+4}(\mathbb{T} \times \mathbb{R}))$.

We decompose Equation (III.117a) into

$$\partial_t q_\varepsilon^R + \mathcal{H}q_\varepsilon^R + \mathcal{B}q_\varepsilon^R + \mathcal{D}q_\varepsilon^R = S_\varepsilon^{0,R,N} - \mathcal{F}v_\varepsilon^R, \quad (\text{III.147})$$

where operators \mathcal{H} , \mathcal{B} , \mathcal{D} and \mathcal{F} are defined as

$$\begin{aligned} \mathcal{H}q_\varepsilon^R &:= \varepsilon^{-1} h \partial_y q_\varepsilon^R, \quad \mathcal{F}v_\varepsilon^R := \varepsilon^{M-2} (v_\varepsilon^{1R} \partial_\theta q_\varepsilon^a + \varepsilon v_\varepsilon^{2R} \partial_y q_\varepsilon^a) + C \varepsilon^{M-2} q_\varepsilon^a (\partial_\theta v_\varepsilon^{1R} + \varepsilon \partial_y v_\varepsilon^{2R}), \\ \mathcal{B}q_\varepsilon^R &:= \varepsilon^{M-2} (v_\varepsilon^{1a} \partial_\theta q_\varepsilon^R + \varepsilon v_\varepsilon^{2a} \partial_y q_\varepsilon^R), \quad \mathcal{D}q_\varepsilon^R := C \varepsilon^{M-2} q_\varepsilon^R (\partial_\theta v_\varepsilon^{1a} + \varepsilon \partial_y v_\varepsilon^{2a}). \end{aligned}$$

The energy method in $H_{(1,\varepsilon)}^m$ consists in derivating Equation (III.126) by $\varepsilon^{\alpha_1} \partial^\alpha$. Then we multiply it by $\varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R$ and integrate (with respect to the space variables (θ, y)),

$$\begin{aligned} \frac{1}{2} \partial_t \|\varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R\|_{L^2}^2 &+ \langle \varepsilon^{\alpha_1} \partial^\alpha (\mathcal{H}q_\varepsilon^R), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle \\ &+ \langle \varepsilon^{\alpha_1} \partial^\alpha (\mathcal{B}q_\varepsilon^R), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle + \langle \varepsilon^{\alpha_1} \partial^\alpha (\mathcal{D}v_\varepsilon^R), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle \\ &= \langle \varepsilon^{\alpha_1} \partial^\alpha S_\varepsilon^{0,R,N}, \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle - \langle \varepsilon^{\alpha_1} \partial^\alpha (\mathcal{F}v_\varepsilon^R), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle. \end{aligned} \quad (\text{III.148})$$

In Subsection 3.2.1, we estimate each of these contributions to prove Lemma III.31 (at the end of Subsection 3.2.1). Then in Subsection 3.2.2, we prove by induction that Proposition III.30 is satisfied.

3.2.1 Inequality of energy for the pressure

To prove Lemma III.31, we state several lemmas where we estimate each terms in Equation (III.148) depending on their structure.

•**Estimates for operators acting on the pressure**• First we exhibit a control over the contribution of operator \mathcal{H} in anisotropic spaces $H_{(1,\varepsilon)}^m$ where it is no longer degenerate. The control was partially obtained in Lemma III.27. Yet, we quickly recover the estimate in $H_{(1,\varepsilon)}^m$. Then we deal with the case of operators \mathcal{B} and \mathcal{D} .

Lemma III.32. *Select a multi-index $\alpha \in \mathbb{N}^2$ of length smaller than $m + 3$. The following holds,*

i) *if $\alpha_1 = 0$, then for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$:*

$$|\langle \partial^\alpha (\varepsilon^{-1} h \partial_y q_\varepsilon^R), \partial^\alpha q_\varepsilon^R \rangle| (t) = 0;$$

ii) *if $\alpha_1 \geq 1$, there exist two positive constants $C_{\mathcal{H}}^1$ and $C_{\mathcal{H}}^2$ such that for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$:*

$$|\langle \varepsilon^{\alpha_1} \partial^\alpha (\varepsilon^{-1} h \partial_y q_\varepsilon^R), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle| (t) \leq C_{\mathcal{H}}^1 \|q_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^{|\alpha|}(\mathbb{T} \times \mathbb{R})}^2 + C_{\mathcal{H}}^2 \|q_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^{|\alpha|-1}(\mathbb{T} \times \mathbb{R})}^2.$$

Proof of Lemma III.32. The first case $i)$ is obvious. We consider $\alpha \in \mathbb{N}^2$ such that $\alpha_1 \geq 1$.

$$\begin{aligned} |\langle \varepsilon^{\alpha_1} \partial^\alpha (\varepsilon^{-1} h \partial_y q_\varepsilon^R), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle| &= |\langle \varepsilon^{\alpha_1} [\partial^\alpha, \varepsilon^{-1} h \partial_y] q_\varepsilon^R, \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle|, \\ &\leq \frac{1}{2} \left(\|\varepsilon^{\alpha_1} [\partial^\alpha, \varepsilon^{-1} h \partial_y] q_\varepsilon^R\|_{L^2}^2 + \|\varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R\|_{L^2}^2 \right). \end{aligned} \quad (\text{III.149})$$

There remains to estimate the commutator. As ∂_y and h commute, the only contribution comes from the derivatives with respect to θ , derivating term by term,

$$\begin{aligned} \|\varepsilon^{\alpha_1} [\partial^\alpha, \varepsilon^{-1} h \partial_y] q_\varepsilon^R\|_{L^2} &= \|\varepsilon^{\alpha_1} [\partial_\theta^{\alpha_1}, \varepsilon^{-1} h \partial_y^{\alpha_2+1}] q_\varepsilon^R\|_{L^2}, \\ &\leq \sum_{k=0}^{\alpha_1-1} C_{\alpha_1}^k \left\| \varepsilon^{\alpha_1-1} \partial_\theta^{\alpha_1-k} h \partial_\theta^k \partial_y^{\alpha_2+1} q_\varepsilon^R \right\|_{L^2}, \\ &\leq \|h\|_{W^{m+3,\infty}} \sum_{k=0}^{\alpha_1-1} C_{\alpha_1}^k \left\| \varepsilon^{\alpha_1-1} \partial_\theta^k \partial_y^{\alpha_2+1} q_\varepsilon^R \right\|_{L^2}^2, \\ &\leq \|h\|_{W^{m+3,\infty}} \sum_{k=0}^{\alpha_1-2} C_{\alpha_1}^k \|q_\varepsilon^R\|_{H_{(1,\varepsilon)}^{|\alpha|-1}} + \|h\|_{W^{m+3,\infty}} \|q_\varepsilon^R\|_{\dot{H}_{(1,\varepsilon)}^{|\alpha|}}. \end{aligned}$$

Elevating to the power two we obtain,

$$\|\varepsilon^{\alpha_1} [\partial^\alpha, \varepsilon^{-1} h \partial_y] q_\varepsilon^R\|_{L^2}^2 \leq 2 \left(\|h\|_{W^{m+3,\infty}} \sum_{k=0}^{\alpha_1-2} C_{\alpha_1}^k \right)^2 \|q_\varepsilon^R\|_{H_{(1,\varepsilon)}^{|\alpha|-1}}^2 + 2 \|h\|_{W^{m+3,\infty}}^2 \|q_\varepsilon^R\|_{\dot{H}_{(1,\varepsilon)}^{|\alpha|}}^2.$$

Plugging it into Equation (III.149) we deduce that for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$,

$$\|\varepsilon^{\alpha_1} [\partial^\alpha, \varepsilon^{-1} h \partial_y] q_\varepsilon^R\|_{L^2}^2(t) \leq C_{\mathcal{H}}^1 \|q_\varepsilon^R(t, \cdot)\|_{\dot{H}_{(1,\varepsilon)}^{|\alpha|}}^2 + C_{\mathcal{H}}^2 \|q_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^{|\alpha|-1}}^2,$$

$$\text{with } C_{\mathcal{H}}^1 := 1/2 + 2 \|h\|_{W^{m+3,\infty}}^2 \text{ and } C_{\mathcal{H}}^2 := 2 \left(\|h\|_{W^{m+3,\infty}} \sum_{k=0}^{\alpha_1-2} C_{\alpha_1}^k \right)^2. \quad \blacksquare$$

We now estimate the contribution of \mathcal{B} and \mathcal{D} in Equation (III.148). The proof is exactly the same as the one performed for Lemma III.24. Nevertheless, as we use anisotropic Sobolev spaces instead of Sobolev spaces, we recall steps of the proof. In particular, we underline the fact that it needs $M \geq 3$.

Lemma III.33. Assume $M \geq 3$. Select a multi-index $\alpha \in \mathbb{N}^2$ with length smaller than $m+3$ the following holds,

i) if $|\alpha| = 0$, there exists a positive constant $C_{\mathcal{B}}^1$ such that for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$,

$$|\langle \mathcal{B} q_\varepsilon^R, q_\varepsilon^R \rangle|(t) \leq C_{\mathcal{B}}^1 \|q_\varepsilon^R(t, \cdot)\|_{L^2(\mathbb{T} \times \mathbb{R})}^2;$$

ii) if $|\alpha| \geq 1$, there exist two positive constants $C_{\mathcal{B}}^1$ and $C_{\mathcal{B}}^2$ such that for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$,

$$|\langle \varepsilon^{\alpha_1} \partial^\alpha (\mathcal{B} q_\varepsilon^R), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle|(t) \leq C_{\mathcal{B}}^1 \|q_\varepsilon^R(t, \cdot)\|_{\dot{H}_{(1,\varepsilon)}^{|\alpha|}(\mathbb{T} \times \mathbb{R})}^2 + C_{\mathcal{B}}^2 \|q_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^{|\alpha|-1}(\mathbb{T} \times \mathbb{R})}^2.$$

Proof of Lemma III.33. We skip the case *i)* i.e. $|\alpha| = 0$. The proof is obvious integrating by parts. To prove *ii)* we consider a multi-index $\alpha \in \mathbb{N}^2$ such that $|\alpha| \geq 1$. We have:

$$\begin{aligned} \langle \varepsilon^{\alpha_1} \partial^\alpha (\mathcal{B} q_\varepsilon^R), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle &= \varepsilon^{M-2} \langle \varepsilon^{\alpha_1} \partial^\alpha (v_\varepsilon^{a_1} \partial_\theta q_\varepsilon^R), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle \\ &\quad + \varepsilon^{M-2} \langle \varepsilon^{\alpha_1} \partial^\alpha (\varepsilon v_\varepsilon^{a_2} \partial_y q_\varepsilon^R), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle. \end{aligned}$$

◦ We start by studying the term $\varepsilon^{M-2} \langle \varepsilon^{\alpha_1} \partial^\alpha (v_\varepsilon^{a_1} \partial_\theta q_\varepsilon^R), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle$. Applying the Leibniz formula, there exists a family $\{C_{\alpha,\beta}\}$ of positive constants such that,

$$\begin{aligned} &\varepsilon^{M-2} |\langle \varepsilon^{\alpha_1} \partial^\alpha (v_\varepsilon^{a_1} \partial_\theta q_\varepsilon^R), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle| \\ &= \varepsilon^{M-2} \sum_{\beta < \alpha} C_{\alpha,\beta} \int_{\mathbb{T} \times \mathbb{R}} \varepsilon^{2\alpha_1} \partial^{\alpha-\beta} (v_\varepsilon^{a_1}) \partial^\beta (\partial_\theta q_\varepsilon^R) \partial^\alpha q_\varepsilon^R d\theta dy \\ &\quad + \varepsilon^{(M-2)} \int_{\mathbb{T} \times \mathbb{R}} \varepsilon^{2\alpha_1} v_\varepsilon^{a_1} \partial^\alpha \partial_\theta (q_\varepsilon^R) \partial^\alpha q_\varepsilon^R d\theta dy, \\ &= \varepsilon^{M-2} \sum_{\beta < \alpha} C_{\alpha,\beta} \int_{\mathbb{T} \times \mathbb{R}} \varepsilon^{2\alpha_1} \partial^{\alpha-\beta} v_\varepsilon^{a_1} \partial^\beta (\partial_\theta q_\varepsilon^R) \partial^\alpha q_\varepsilon^R d\theta dy \\ &\quad - \varepsilon^{(M-2)/2} \int_{\mathbb{T} \times \mathbb{R}} \partial_\theta v_\varepsilon^{a_1} (\varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R)^2 d\theta dy. \quad (\text{III.150}) \end{aligned}$$

-The last term in Equation (III.150) is rather easy to estimate, for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\begin{aligned} \left| -\varepsilon^{(M-2)/2} \int_{\mathbb{T} \times \mathbb{R}} \partial_\theta v_\varepsilon^{a_1}(t, \cdot) (\varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R(t, \cdot))^2 d\theta dy \right| &\leq \|v_\varepsilon^a(t, \cdot)\|_{W^{1,\infty}} / 2 \|q_\varepsilon(t, \cdot)\|_{H_{(1,\varepsilon)}^{|\alpha|}}^2, \\ &\leq C_a / 2 \|q_\varepsilon(t, \cdot)\|_{H_{(1,\varepsilon)}^{|\alpha|}}^2. \end{aligned}$$

-There remains to compute the sum in Equation (III.150). This is one of the problematic term. Indeed, it appears the family $\{\partial_\theta q_\varepsilon^R\}_\varepsilon$ which is not bounded in L^2 . We make appear the ε -derivative $\varepsilon \partial_\theta$ paying a loss of precision on M . For all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$,

$$\begin{aligned} &\left| \varepsilon^{M-2} \int_{\mathbb{T} \times \mathbb{R}} \varepsilon^{2\alpha_1} \partial^{\alpha-\beta} v_\varepsilon^{a_1}(t, \cdot) \partial^\beta \partial_\theta q_\varepsilon^R(t, \cdot) \partial^\alpha q_\varepsilon^R(t, \cdot) d\theta dy \right| \\ &= \left| \varepsilon^{M-3} \int_{\mathbb{T} \times \mathbb{R}} \varepsilon^{2\alpha_1+1} \partial^{\alpha-\beta} v_\varepsilon^{a_1}(t, \cdot) \partial^\beta \partial_\theta q_\varepsilon^R(t, \cdot) \partial^\alpha q_\varepsilon^R(t, \cdot) d\theta dy \right|, \\ &\leq \frac{\|v_\varepsilon^{a_1}(t, \cdot)\|_{W^{m+4,\infty}}}{2} \left(\left\| \varepsilon^{\alpha_1+1} \partial^\beta \partial_\theta q_\varepsilon^R(t, \cdot) \right\|_{L^2}^2 + \left\| \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R(t, \cdot) \right\|_{L^2}^2 \right), \\ &\leq \frac{C_a}{2} \left(\|q_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^{|\alpha|-1}}^2 + 2 \|q_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^{|\alpha|}}^2 \right). \end{aligned}$$

Thus if $M \geq 3$, there exist $A := \sum_{\beta < \alpha} C_{\alpha,\beta} C_a / 2 > 0$ and $B := (\sum_{\beta < \alpha} C_{\alpha,\beta} + 1/2) C_a > 0$ such that for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$,

$$\varepsilon^{M-2} |\langle \varepsilon^{\alpha_1} \partial^\alpha (v_\varepsilon^{a_1} \partial_\theta q_\varepsilon^R), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle| (t) \leq A \|q_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^{|\alpha|-1}}^2 + B \|q_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^{|\alpha|}}^2. \quad (\text{III.151})$$

◦ Of course we can perform the same estimate for $\varepsilon^{M-2} \langle \varepsilon^{\alpha_1} \partial^\alpha (\varepsilon v_\varepsilon^{a_2} \partial_y q_\varepsilon^R, \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R) \rangle$. However it only requires $M \geq 2$ (It appears the derivative ∂_y instead of ∂_θ). There exist two positive constants C and D such that for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$|\varepsilon^{M-2} \langle \varepsilon^{\alpha_1} \partial^\alpha (\varepsilon v_\varepsilon^{a_2} \partial_y q_\varepsilon^R), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle| (t) \leq C \|q_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^{|\alpha|-1}}^2 + D \|q_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^{|\alpha|}}^2. \quad (\text{III.152})$$

◦ Putting Inequality (III.151) and (III.152) together it leads that for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$|\langle \partial^\alpha (\mathcal{B} q_\varepsilon^R), \partial^\alpha q_\varepsilon^R \rangle| \leq (A + C) \|q_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^{|\alpha|-1}}^2 + (B + D) \|q_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^{|\alpha|}}^2.$$

■

The same estimates work for the contribution of \mathcal{D} .

Lemma III.34. *Assume $M \geq 3$. Select a multi-index $\alpha \in \mathbb{N}^2$ with length smaller than $m + 3$. The followings hold.*

i) *If $|\alpha| = 0$, there exists a positive constant $C_{\mathcal{D}}^1$ such that for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$:*

$$|\langle \mathcal{D} q_\varepsilon^R, q_\varepsilon^R \rangle| (t) \leq C_{\mathcal{D}}^1 \|q_\varepsilon^R(t, \cdot)\|_{L^2(\mathbb{T} \times \mathbb{R})}^2.$$

ii) *If $|\alpha| \geq 1$, there exist two positive constants $C_{\mathcal{D}}^1$ and $C_{\mathcal{D}}^2$ such that for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$:*

$$|\langle \varepsilon^{\alpha_1} \partial^\alpha (\mathcal{D} q_\varepsilon^R), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle| (t) \leq C_{\mathcal{D}}^1 \|q_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^{|\alpha|}(\mathbb{T} \times \mathbb{R})}^2 + C_{\mathcal{D}}^2 \|q_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^{|\alpha|-1}(\mathbb{T} \times \mathbb{R})}^2.$$

• **Estimate for operator \mathcal{F}** • We now estimate the operator \mathcal{F} . The main difference with the three previous operators is that \mathcal{F} is an operator which acts on the velocity v_ε^R . It is hoped to be rather small recalling that the velocity v_ε^R does satisfy (III.122) and (III.123). Nonetheless, we have to be prudent. Terms of the form

$$\partial^\alpha (q_\varepsilon^a (\partial_\theta v_\varepsilon^{1R} + \partial_y v_\varepsilon^{2R}))$$

are singular (with respect to the number of derivatives) in L^2 -norm. Thus it requires a control of the velocity in H^{m+4} .

Lemma III.35. *Assume $M \geq 7/2$. Select a multi-index $\alpha \in \mathbb{N}^2$ with length smaller than $m + 3$. There exist two positive constants $C_{\mathcal{F}}^1$ and $S_{\mathcal{F}}$ such that for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$:*

$$|\langle \varepsilon^{\alpha_1} \partial^\alpha (\mathcal{F} v_\varepsilon^R), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle| (t) \leq S_{\mathcal{F}} \|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}(\mathbb{T} \times \mathbb{R})}^2 + C_{\mathcal{F}}^1 \|q_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^{|\alpha|}(\mathbb{T} \times \mathbb{R})}^2.$$

Proof of Lemma III.35. Select a multi-index $\alpha \in \mathbb{N}^2$ satisfying $|\alpha| \leq m+3$. We decompose \mathcal{F} as follows:

$$\begin{aligned} \langle \varepsilon^{\alpha_1} \partial^\alpha (\mathcal{F} q_\varepsilon^R), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle &= \langle \varepsilon^{M-2} \varepsilon^{\alpha_1} \partial^\alpha (v_\varepsilon^{1R} \partial_\theta q_\varepsilon^a), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle \\ &\quad + \langle \varepsilon^{M-2} \varepsilon^{\alpha_1} \partial^\alpha (\varepsilon v_\varepsilon^{2R} \partial_y q_\varepsilon^a), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle \\ &\quad + \langle C \varepsilon^{M-2} \varepsilon^{\alpha_1} \partial^\alpha (q_\varepsilon^a \partial_\theta v_\varepsilon^{1R}), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle + \langle C \varepsilon^{M-2} \varepsilon^{\alpha_1} \partial^\alpha (q_\varepsilon^a \varepsilon \partial_y v_\varepsilon^{2R}), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle. \end{aligned} \quad (\text{III.153})$$

◦ We start to estimate the contribution of $\langle \varepsilon^{M-2} \varepsilon^{\alpha_1} \partial^\alpha (v_\varepsilon^{1R} \partial_\theta q_\varepsilon^a), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle$:

$$\langle \varepsilon^{M-2} \varepsilon^{\alpha_1} \partial^\alpha (v_\varepsilon^{1R} \partial_\theta q_\varepsilon^a), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle \leq \frac{1}{2} \left(\left\| \varepsilon^{M-2} \varepsilon^{\alpha_1} \partial^\alpha (v_\varepsilon^{1R} \partial_\theta q_\varepsilon^a) \right\|_{L^2}^2 + \left\| \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \right\|_{L^2}^2 \right)$$

To control $\left\| \varepsilon^{M-2} \varepsilon^{\alpha_1} \partial^\alpha (v_\varepsilon^{1R} \partial_\theta q_\varepsilon^a) \right\|$ we could apply the Gagliardo-Nirenberg inequality. We would have to control term of the form $\left\| \varepsilon^{\alpha_1} \partial_\theta q_\varepsilon^a \right\|_{\dot{H}^{|\alpha|}}$. Using the equivalence of norms between the anisotropic Sobolev Space and the Sobolev space (III.106) we may lose power of ε up to order $|\alpha|$. Here a simple computation is more accurate. Indeed, using the Leibniz formula there exists a family of positive constant $\{C_{\alpha,\beta}\}$ such that $\partial^\alpha (v_\varepsilon^{1R} \partial_\theta q_\varepsilon^a) = \sum_{0 \leq \beta \leq \alpha} C_{\alpha,\beta} \partial^{\alpha-\beta} v_\varepsilon^{1R} \partial^\beta \partial_\theta q_\varepsilon^a$. Then apply the Minkovski inequality

$$\begin{aligned} \left\| \varepsilon^{M-2} \varepsilon^{\alpha_1} \partial^\alpha (v_\varepsilon^{1R} \partial_\theta q_\varepsilon^a) \right\|_{L^2} &\leq \varepsilon^{M-2} \sum_{0 \leq \beta \leq \alpha} C_{\alpha,\beta} \left\| \partial^{\alpha-\beta} v_\varepsilon^{1R} \varepsilon^{\alpha_1} \partial^\beta \partial_\theta q_\varepsilon^a \right\|_{L^2}, \\ &\leq \varepsilon^{M-3} \sum_{0 \leq \beta \leq \alpha} C_{\alpha,\beta} \left\| \varepsilon^{\alpha_1+1} \partial^\beta \partial_\theta q_\varepsilon^a \right\|_{L^\infty} \left\| \partial^{\alpha-\beta} v_\varepsilon^{1R} \right\|_{L^2}, \\ &\leq \varepsilon^{M-3} \sum_{0 \leq \beta \leq \alpha} C_{\alpha,\beta} \|q_\varepsilon^a\|_{W_{(1,\varepsilon)}^{m+4,\infty}} \|v_\varepsilon^R\|_{H^{m+3}}, \\ &\leq \boxed{\varepsilon^{M-7/2}} \sum_{0 \leq \beta \leq \alpha} C_{\alpha,\beta} \|q_\varepsilon^a\|_{H_{(1,\varepsilon)}^{m+6}} \|v_\varepsilon^R\|_{H^{m+3}}. \end{aligned}$$

In the above inequality, we use the following imbeddings $\|q_\varepsilon^a(t, \cdot)\|_{W_{(1,\varepsilon)}^{1,\infty}} \leq \varepsilon^{-\frac{1}{2}} \|q_\varepsilon^a(t, \cdot)\|_{H_{(1,\varepsilon)}^{|\alpha|+1}}$.

As we assume $M \geq 7/2$ we get,

$$\left\| \varepsilon^{M-2} \varepsilon^{\alpha_1} \partial^\alpha (v_\varepsilon^{1R} \partial_\theta q_\varepsilon^a) \right\|_{L^2} \leq C_a \sum_{0 \leq \beta \leq \alpha} C_{\alpha,\beta} \|v_\varepsilon^R(t, \cdot)\|_{H^{m+3}}.$$

Finally for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$,

$$\begin{aligned} &|\langle \varepsilon^{M-2} \varepsilon^{\alpha_1} \partial^\alpha (v_\varepsilon^{1R} \partial_\theta q_\varepsilon^a), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle|(t) \\ &\leq \frac{1}{2} \left(C_a^2 \left(\sum_{0 \leq \beta \leq \alpha} C_{\alpha,\beta} \right)^2 \|v_\varepsilon^R(t, \cdot)\|_{H^{m+3}}^2 + \left\| \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R(t, \cdot) \right\|_{L^2}^2 \right). \end{aligned} \quad (\text{III.154})$$

◦ Performing the same estimate, we can deal with the term $\langle \varepsilon^{M-2} \varepsilon^{\alpha_1} \partial^\alpha (\varepsilon v_\varepsilon^{2R} \partial_y q_\varepsilon^a), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle$,

i.e. for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$,

$$\begin{aligned} & |\langle \varepsilon^{M-2} \varepsilon^{\alpha_1} \partial^\alpha (\varepsilon v_\varepsilon^{2R} \partial_y q_\varepsilon^a), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle|(t) \\ & \leq \frac{1}{2} \left(C_a^2 \left(\sum_{0 \leq \beta \leq \alpha} C_{\alpha, \beta} \right)^2 \|v_\varepsilon^R(t, \cdot)\|_{H^{m+3}}^2 + \|\varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R(t, \cdot)\|_{L^2}^2 \right). \end{aligned} \quad (\text{III.155})$$

◦ We now move to the estimate of $\langle C \varepsilon^{M-2} \varepsilon^{\alpha_1} \partial^\alpha (q_\varepsilon^a \partial_\theta v_\varepsilon^{1R}), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle$. First, apply the Cauchy-Schwarz inequality together with the Young inequality:

$$|\langle C \varepsilon^{M-2} \varepsilon^{\alpha_1} \partial^\alpha (q_\varepsilon^a \partial_\theta v_\varepsilon^{1R}), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle| \leq \frac{1}{2} \left(\|C \varepsilon^{M-2} \varepsilon^{\alpha_1} \partial^\alpha (q_\varepsilon^a \partial_\theta v_\varepsilon^{1R})\|_{L^2}^2 + \|\varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R\|_{L^2}^2 \right).$$

To get rid with the contribution of the nonlinear term $\|C \varepsilon^{M-2} \varepsilon^{\alpha_1} \partial^\alpha (q_\varepsilon^a \partial_\theta v_\varepsilon^{1R})\|_{L^2}$ we start with a Leibniz formula, then we apply the Minkovski inequality:

$$\begin{aligned} \|C \varepsilon^{M-2} \varepsilon^{\alpha_1} \partial^\alpha (q_\varepsilon^a \partial_\theta v_\varepsilon^{1R})\|_{L^2} & \leq C \varepsilon^{M-2} \sum_{0 \leq \beta \leq \alpha} C_{\alpha, \beta} \left\| \varepsilon^{\alpha_1} \partial^{\alpha-\beta} \partial_\theta v_\varepsilon^{1R} \partial^\beta q_\varepsilon^a \right\|_{L^2}, \\ & \leq C \varepsilon^{M-2} \sum_{0 \leq \beta \leq \alpha} C_{\alpha, \beta} \left\| \varepsilon^{\alpha_1} \partial^\beta q_\varepsilon^a \right\|_{L^\infty} \left\| \partial^{\alpha-\beta} \partial_\theta v_\varepsilon^{1R} \right\|_{L^2}, \\ & \leq C \varepsilon^{M-2} \sum_{0 \leq \beta \leq \alpha} C_{\alpha, \beta} \|q_\varepsilon^a\|_{W^{|\alpha|, \infty}_{(1, \varepsilon)}} \|v_\varepsilon^R\|_{H^{m+4}}, \\ & \leq C \boxed{\varepsilon^{M-5/2}} \sum_{0 \leq \beta \leq \alpha} C_{\alpha, \beta} \|q_\varepsilon^a\|_{H^{m+5}_{(1, \varepsilon)}} \|v_\varepsilon^R\|_{H^{m+4}}, \\ & \leq C C_a \sum_{0 \leq \beta \leq \alpha} C_{\alpha, \beta} \|v_\varepsilon^R\|_{H^{m+4}}. \end{aligned}$$

So for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$ we have,

$$\begin{aligned} & |\langle C \varepsilon^{M-2} \varepsilon^{\alpha_1} \partial^\alpha (q_\varepsilon^a \partial_\theta v_\varepsilon^{1R}), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle| \\ & \leq \frac{1}{2} C^2 C_a^2 \left(\sum_{0 \leq \beta \leq \alpha} C_{\alpha, \beta} \right)^2 \|v_\varepsilon^R\|_{H^{m+4}}^2 + \frac{1}{2} \|q_\varepsilon^R\|_{\dot{H}^{|\alpha|}_{(1, \varepsilon)}}^2. \end{aligned} \quad (\text{III.156})$$

◦ The same calculation leads to for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\begin{aligned} & |\langle C \varepsilon^{M-2} \varepsilon^{\alpha_1} \partial^\alpha (q_\varepsilon^a \varepsilon \partial_y v_\varepsilon^{2R}), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle| \\ & \leq \frac{1}{2} C^2 C_a^2 \left(\sum_{0 \leq \beta \leq \alpha} C_{\alpha, \beta} \right)^2 \|v_\varepsilon^R\|_{H^{m+4}}^2 + \frac{1}{2} \|q_\varepsilon^R\|_{\dot{H}^{|\alpha|}_{(1, \varepsilon)}}^2. \end{aligned} \quad (\text{III.157})$$

Plugging (III.154), (III.155), (III.156) and (III.157) into (III.153), it proves Lemma III.35. ■

• **Estimate of the source term $S_\varepsilon^{0, R, N}$** • The term $S_\varepsilon^{0, R, N}$ contains all difficulties ever met. First, there are non-linear terms involving derivatives of the pressure q_ε^R . We again perform

an integration by parts to deal with those contributions. Secondly, we control the singular term (with respect to ε) $\partial_\theta q_\varepsilon^R$ in L^∞ -norm to a cost on the the integer M and then on the precision w_m .

Lemma III.36. *Assume $M \geq 7/2$. Let a multi-index $\alpha \in \mathbb{N}^2$ with length smaller than $m+3$. It holds*

i) if $|\alpha| = 0$, there exist two positive constants C_S^1 and S_S such that for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\begin{aligned} |\langle S_\varepsilon^{0,R,N}, q_\varepsilon^R \rangle| (t) &\leq C_S^1 \left(1 + \|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}(\mathbb{T} \times \mathbb{R})}^2 \right) \|q_\varepsilon^R(t, \cdot)\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 \\ &\quad + S_S \left(\|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}(\mathbb{T} \times \mathbb{R})}^2 + \varepsilon^{2(N-R)} \right); \end{aligned}$$

ii) if $|\alpha| \geq 1$, there exist three positive constants C_S^1 , C_S^2 and S_S such that for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\begin{aligned} |\langle \varepsilon^{\alpha_1} \partial^\alpha S_\varepsilon^{0,R,N}, \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle| (t) &\leq C_S^1 \left(1 + \|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}(\mathbb{T} \times \mathbb{R})}^2 \right) \|q_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^{|\alpha|}(\mathbb{T} \times \mathbb{R})}^2 \\ &\quad + C_S^2 \left(1 + \|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}(\mathbb{T} \times \mathbb{R})}^2 \right) \|q_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^{|\alpha|-1}(\mathbb{T} \times \mathbb{R})}^2 \\ &\quad + S_S \left(\|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}(\mathbb{T} \times \mathbb{R})}^2 + \varepsilon^{2(N-R)} \right). \end{aligned}$$

Proof of Lemma III.36. We only prove the result for $|\alpha| \geq 1$. The case $|\alpha| = 0$ can be dealt performing the same proof. The first step is to decompose $S_\varepsilon^{0,R,N}$ depending on the structure of the terms :

$$\begin{aligned} \langle \varepsilon^{\alpha_1} \partial^\alpha S_\varepsilon^{0,R,N}, \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle &= \langle \varepsilon^{\alpha_1} \partial^\alpha \varepsilon^{N-R} (\varepsilon^{-N} \mathcal{L}_0(\varepsilon, q_\varepsilon^a, v_\varepsilon^a)), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle \\ &\quad + \langle \varepsilon^{R+M-2} \varepsilon^{\alpha_1} \partial^\alpha (v_\varepsilon^{1R} \partial_\theta q_\varepsilon^R), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle + \langle \varepsilon^{R+M-2} \varepsilon^{\alpha_1} \partial^\alpha (v_\varepsilon^{2R} \partial_y q_\varepsilon^R), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle \\ &\quad + \langle C \varepsilon^{R+M-2} \varepsilon^{\alpha_1} \partial^\alpha (q_\varepsilon^R \partial_\theta v_\varepsilon^{1R}), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle + \langle C \varepsilon^{R+M-2} \varepsilon^{\alpha_1} \partial^\alpha (q_\varepsilon^R \partial_y v_\varepsilon^{2R}), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle. \end{aligned}$$

In what follows we study each of these contributions.

◦ *Contribution of the First term.* Performing a Cauchy-Schwarz estimate together with the Young inequality, for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\begin{aligned} |\langle \varepsilon^{\alpha_1} \partial^\alpha \varepsilon^{N-R} (\varepsilon^{-N} \mathcal{L}_0(\varepsilon, q_\varepsilon^a, v_\varepsilon^a)), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle| (t) &\leq \frac{1}{2} \left(\varepsilon^{2(N-R)} \|\varepsilon^{-N} \mathcal{L}_0(\varepsilon, q_\varepsilon^a, v_\varepsilon^a)\|_{H_{(1,\varepsilon)}^{|\alpha|}}^2 + \|q_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^{|\alpha|}}^2 \right), \\ &\leq \frac{1}{2} \left(\varepsilon^{2(N-R)} C_{\mathcal{L}}^2 + \|q_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^{|\alpha|}}^2 \right). \end{aligned}$$

◦ *Contribution of the second term.* We now estimate the contribution of

$$\langle \varepsilon^{R+M-2} \varepsilon^{\alpha_1} \partial^\alpha (v_\varepsilon^{1R} \partial_\theta q_\varepsilon^R), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle.$$

We expand it thanks to the Leibniz formula. There exists a family of positive constant $\{C_{\alpha,\beta}\}$

such that

$$\begin{aligned}
& \left| \left\langle \varepsilon^{R+M-2} \varepsilon^{\alpha_1} \partial^\alpha \left(v_\varepsilon^{1R} \partial_\theta q_\varepsilon^R \right), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \right\rangle \right| \\
&= \varepsilon^{R+M-2} \sum_{0 \leq \beta \leq \alpha} C_{\alpha, \beta} \int_{\mathbb{T} \times \mathbb{R}} \varepsilon^{2\alpha_1} \partial^{\alpha-\beta} v_\varepsilon^{1R} \partial_\theta \partial^\beta q_\varepsilon^R \partial^\alpha q_\varepsilon^R d\theta dy, \\
&= \varepsilon^{R+M-2} \int_{\mathbb{T} \times \mathbb{R}} \varepsilon^{2\alpha_1} \partial^\alpha v_\varepsilon^{1R} \partial_\theta q_\varepsilon^R \partial^\alpha q_\varepsilon^R d\theta dy \\
&\quad + \varepsilon^{R+M-2} \sum_{0 < \beta < \alpha} C_{\alpha, \beta} \int_{\mathbb{T} \times \mathbb{R}} \varepsilon^{2\alpha_1} \partial^{\alpha-\beta} v_\varepsilon^{1R} \partial_\theta \partial^\beta q_\varepsilon^R \partial^\alpha q_\varepsilon^R d\theta dy \\
&\quad - \varepsilon^{R+M-2} \int_{\mathbb{T} \times \mathbb{R}} \varepsilon^{2\alpha_1} \frac{\partial_\theta v_\varepsilon^{1R}}{2} (\partial^\alpha q_\varepsilon^R)^2 d\theta dy. \quad (\text{III.158})
\end{aligned}$$

-First, we deal with $-\varepsilon^{R+M-2} \int_{\mathbb{T} \times \mathbb{R}} \varepsilon^{2\alpha_1} \frac{\partial_\theta v_\varepsilon^{1R}}{2} (\partial^\alpha q_\varepsilon^R)^2 d\theta dy$. By bounding, for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$,

$$\begin{aligned}
& \left| \varepsilon^{R+M-2} \int_{\mathbb{T} \times \mathbb{R}} \varepsilon^{2\alpha_1} \frac{\partial_\theta v_\varepsilon^{1R}}{2} (\partial^\alpha q_\varepsilon^R)^2 d\theta dy \right| \\
&\leq \frac{1}{2} \|\partial_\theta v_\varepsilon^R\|_{L^\infty} \|\varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R\|_{L^2}^2 \leq \|q_\varepsilon^R\|_{\dot{H}_{(1, \varepsilon)}^{|\alpha|}}^2. \quad (\text{III.159})
\end{aligned}$$

-Then, we deal with the second term $\varepsilon^{R+M-2} \int_{\mathbb{T} \times \mathbb{R}} \varepsilon^{2\alpha_1} \partial^\alpha v_\varepsilon^{1R} \partial_\theta q_\varepsilon^R \partial^\alpha q_\varepsilon^R d\theta dy$. This term is more problematic as it contains the derivative ∂_θ of the pressure. As ever mentionned, we have to introduce a power of ε (thanks to M) to get rid of the derivative with respect to θ , ∂_θ .

$$\begin{aligned}
& \varepsilon^{R+M-2} \left| \left\langle \varepsilon^{2\alpha_1} \partial^\alpha v_\varepsilon^{1R}(t, \cdot) \partial_\theta q_\varepsilon^R(t, \cdot), \partial^\alpha q_\varepsilon^R(t, \cdot) \right\rangle \right| \\
&\leq \varepsilon^{R+M-2} \|\partial_\theta q_\varepsilon^R(t, \cdot)\|_{L^\infty} \left| \left\langle \varepsilon^{\alpha_1} \partial^\alpha v_\varepsilon^{1R}(t, \cdot), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R(t, \cdot) \right\rangle \right|, \\
&\leq \frac{\varepsilon^{R+M-2}}{2} \|\partial_\theta q_\varepsilon^R(t, \cdot)\|_{L^\infty} \left(\|v_\varepsilon^R(t, \cdot)\|_{H^{m+3}}^2 + \|q_\varepsilon^R(t, \cdot)\|_{\dot{H}_{(1, \varepsilon)}^{|\alpha|}}^2 \right), \\
&\leq \frac{\varepsilon^{R+M-3}}{2} \|q_\varepsilon^R(t, \cdot)\|_{W_{(1, \varepsilon)}^{1, \infty}} \left(\|v_\varepsilon^R(t, \cdot)\|_{H^{m+3}}^2 + \|q_\varepsilon^R(t, \cdot)\|_{\dot{H}_{(1, \varepsilon)}^{|\alpha|}}^2 \right), \\
&\leq \frac{\varepsilon^{R+M-7/2}}{2} \|q_\varepsilon^R(t, \cdot)\|_{H_{(1, \varepsilon)}^1} \left(\|v_\varepsilon^R(t, \cdot)\|_{H^{m+3}}^2 + \|q_\varepsilon^R(t, \cdot)\|_{\dot{H}_{(1, \varepsilon)}^{|\alpha|}}^2 \right), \\
&\leq \left(\|v_\varepsilon^R(t, \cdot)\|_{H^{m+3}}^2 + \|q_\varepsilon^R(t, \cdot)\|_{\dot{H}_{(1, \varepsilon)}^{|\alpha|}}^2 \right). \quad (\text{III.160})
\end{aligned}$$

Here we crucially use the assumption $M \geq 7/2$.

-There remains to estimate terms which appear in the sum (III.158). First we have,

$$\begin{aligned}
& \varepsilon^{M+R-2} \left| \left\langle \varepsilon^{2\alpha_1} \partial^{\alpha-\beta} v_\varepsilon^{1R} \partial_\theta \partial^\beta q_\varepsilon^R, \partial^\alpha q_\varepsilon^R \right\rangle \right| \\
&\leq \frac{1}{2} \left(\varepsilon^{2(R+M-3)} \left\| \varepsilon^{\alpha_1+1} \partial^{\alpha-\beta} v_\varepsilon^{1R} \partial_\theta \partial^\beta q_\varepsilon^R \right\|_{L^2}^2 + \|\varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R\|_{L^2}^2 \right). \quad (\text{III.161})
\end{aligned}$$

Then we notice that since $0 < \beta < \alpha$, we have $|\alpha - \beta| \leq m + 2$. Thus $\partial^{\alpha-\beta}$ can be bounded in L^∞ . For all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$,

$$\varepsilon^{2(R+M-3)} \left\| \varepsilon^{\alpha+1} \partial^{\alpha-\beta} v_\varepsilon^{1R}(t, \cdot) \partial_\theta \partial^\beta q_\varepsilon^R(t, \cdot) \right\|_{L^2}^2 \leq \|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}}^2 \|q_\varepsilon^R(t, \cdot)\|_{H^{|\alpha|}_{(1,\varepsilon)}}^2.$$

-Finally plugging estimates (III.159), (III.160) and (III.161) into (III.158) we get for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$,

$$\begin{aligned} & \left| \langle \varepsilon^{R+M-2} \varepsilon^{\alpha_1} \partial^\alpha (v_\varepsilon^{1R} \partial_\theta q_\varepsilon^R), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle (t) \right| \\ & \leq \left(\sum_{0 < \beta < \alpha} \frac{C_{\alpha,\beta}}{2} \left(\|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}}^2 + 1 \right) + \frac{3}{2} \right) \|q_\varepsilon^R(t, \cdot)\|_{H^{|\alpha|}_{(1,\varepsilon)}}^2 + \|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}}^2. \end{aligned}$$

◦ *Contribution of the third term.* Of course using the duplicate proof we can deal with the third term, for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\begin{aligned} & \left| \langle \varepsilon^{R+M-2} \varepsilon^{\alpha_1} \partial^\alpha (v_\varepsilon^{2R} \partial_y q_\varepsilon^R), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle (t) \right| \\ & \leq \left(\sum_{\beta < \alpha} \frac{C_{\alpha,\beta}}{2} \left(\|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}}^2 + 1 \right) + \frac{3}{2} \right) \|q_\varepsilon^R(t, \cdot)\|_{H^{|\alpha|}_{(1,\varepsilon)}}^2 + \|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}}^2. \end{aligned}$$

◦ *Contribution of the fourth term.* To estimate $\langle C \varepsilon^{R+M-2} \varepsilon^{\alpha_1} \partial^\alpha (q_\varepsilon^R \partial_\theta v_\varepsilon^{1R}), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle$, we first apply the Cauchy-Schwarz inequality,

$$\left| \langle C \varepsilon^{R+M-2} \varepsilon^{\alpha_1} \partial^\alpha (q_\varepsilon^R \partial_\theta v_\varepsilon^{1R}), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle \right| \leq \frac{1}{2} \left(\|C \varepsilon^{R+M-2} \varepsilon^{\alpha_1} \partial^\alpha (q_\varepsilon^R \partial_\theta v_\varepsilon^{1R})\|_{L^2}^2 + \|q_\varepsilon^R\|_{H^{|\alpha|}_{(1,\varepsilon)}}^2 \right).$$

We still do not apply the Gagliardo-Nirenberg inequality to majorate the nonlinear term $C \varepsilon^{R+M-2} \varepsilon^{\alpha_1} \partial^\alpha (q_\varepsilon^R \partial_\theta v_\varepsilon^{1R})$. Applying the Leibniz formula and the Minkovski inequality, we obtain

$$\|C \varepsilon^{R+M-2} \varepsilon^{\alpha_1} \partial^\alpha (q_\varepsilon^R \partial_\theta v_\varepsilon^{1R})\|_{L^2} \leq C \varepsilon^{R+M-2} \sum_{0 \leq \beta \leq \alpha} C_{\alpha,\beta} \left\| \varepsilon^{\alpha_1} \partial^\beta q_\varepsilon^R \partial^{\alpha-\beta} \partial_\theta v_\varepsilon^{1R} \right\|_{L^2}.$$

Here, we cut the sum in two parts depending on the length of β . When $|\beta|$ is small enough, $\partial^\beta q_\varepsilon^R$ is bounded in L^∞ whereas when $|\beta|$ is large it is $\partial^{\alpha-\beta} \partial_\theta v_\varepsilon^{1R}$ which is bounded in L^∞ . Therefore we decompose the sum into

$$\begin{aligned} \|C \varepsilon^{R+M-2} \varepsilon^{\alpha_1} \partial^\alpha (q_\varepsilon^R \partial_\theta v_\varepsilon^{1R})\|_{L^2} & \leq C \varepsilon^{R+M-2} \sum_{0 \leq \beta \leq \alpha, 0 \leq |\beta| \leq 1} C_{\alpha,\beta} \left\| \varepsilon^{\alpha_1} \partial^\beta q_\varepsilon^R \partial^{\alpha-\beta} \partial_\theta v_\varepsilon^{1R} \right\|_{L^2} \\ & \quad + C \varepsilon^{R+M-2} \sum_{0 \leq \beta \leq \alpha, 2 \leq |\beta| \leq |\alpha|} C_{\alpha,\beta} \left\| \varepsilon^{\alpha_1} \partial^\beta q_\varepsilon^R \partial^{\alpha-\beta} \partial_\theta v_\varepsilon^{1R} \right\|_{L^2}. \end{aligned}$$

We begin estimating the first sum,

$$\begin{aligned}
C\varepsilon^{R+M-2} \sum_{0 \leq \beta \leq \alpha, 0 \leq |\beta| \leq 1} C_{\alpha,\beta} \left\| \varepsilon^{\alpha_1} \partial^\beta q_\varepsilon^R \partial^{\alpha-\beta} \partial_\theta v_\varepsilon^{1R} \right\|_{L^2} \\
\leq C\varepsilon^{R+M-2} \sum_{0 \leq \beta \leq \alpha, 0 \leq |\beta| \leq 1} C_{\alpha,\beta} \left\| \varepsilon^{\alpha_1} \partial^\beta q_\varepsilon^R \right\|_{L^\infty} \left\| \partial^{\alpha-\beta} \partial_\theta v_\varepsilon^{1R} \right\|_{L^2}, \\
\leq C\varepsilon^{R+M-2} \sum_{0 \leq \beta \leq \alpha, 0 \leq |\beta| \leq 1} C_{\alpha,\beta} \|q_\varepsilon^R\|_{W_{(1,\varepsilon)}^{1,\infty}} \|v_\varepsilon^{1R}\|_{H^{m+4}}, \\
\leq C\varepsilon^{R+M-5/2} \sum_{0 \leq \beta \leq \alpha, 0 \leq |\beta| \leq 1} C_{\alpha,\beta} \|q_\varepsilon^R\|_{H_{(1,\varepsilon)}^1} \|v_\varepsilon^{1R}\|_{H^{m+4}}, \\
\leq 2C \boxed{\varepsilon^{R+M-5/2}} \sum_{0 \leq \beta \leq \alpha, 0 \leq |\beta| \leq 1} C_{\alpha,\beta} \|v_\varepsilon^R\|_{H^{m+4}}. \quad (\text{III.162})
\end{aligned}$$

Presently, we majorate the second term by

$$\begin{aligned}
C\varepsilon^{R+M-2} \sum_{0 \leq \beta \leq \alpha, 2 \leq |\beta| \leq |\alpha|} C_{\alpha,\beta} \left\| \varepsilon^{\alpha_1} \partial^\beta q_\varepsilon^R \partial^{\alpha-\beta} \partial_\theta v_\varepsilon^{1R} \right\|_{L^2} \\
\leq C\varepsilon^{R+M-2} \sum_{0 \leq \beta \leq \alpha, 2 \leq |\beta| \leq |\alpha|} C_{\alpha,\beta} \left\| \varepsilon^{\alpha_1} \partial^\beta q_\varepsilon^R \right\|_{L^2} \left\| \partial^{\alpha-\beta} \partial_\theta v_\varepsilon^{1R} \right\|_{L^\infty} \\
\leq C\varepsilon^{R+M-2} \sum_{0 \leq \beta \leq \alpha, 2 \leq |\beta| \leq |\alpha|} C_{\alpha,\beta} \|q_\varepsilon^R\|_{H_{(1,\varepsilon)}^{|\alpha|}} \|v_\varepsilon^R\|_{W^{m+2,\infty}} \\
\leq C\varepsilon^{R+M-2} \sum_{0 \leq \beta \leq \alpha, 2 \leq |\beta| \leq |\alpha|} C_{\alpha,\beta} \|v_\varepsilon^R\|_{H^{m+4}} \|q_\varepsilon^R\|_{H_{(1,\varepsilon)}^{|\alpha|}}. \quad (\text{III.163})
\end{aligned}$$

Joining the two estimates (III.162) and (III.163) together we deduce that for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\begin{aligned}
\|C\varepsilon^{R+M-2} \varepsilon^{\alpha_1} \partial^\alpha (q_\varepsilon^R \partial_\theta v_\varepsilon^{1R})\|_{L^2}^2 &\leq 4C^2 \left(\sum_{0 \leq \beta \leq \alpha, 0 \leq |\beta| \leq 1} C_{\alpha,\beta} \right)^2 \|v_\varepsilon^R\|_{H^{m+4}}^2 \\
&\quad + C^2 \left(\sum_{0 \leq \beta \leq \alpha, 2 \leq |\beta| \leq |\alpha|} C_{\alpha,\beta} \right)^2 \|v_\varepsilon^R\|_{H^{m+4}}^2 \|q_\varepsilon^R\|_{H_{(1,\varepsilon)}^{|\alpha|}}^2.
\end{aligned}$$

Finally putting everything together we obtain that for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\begin{aligned}
|\langle C\varepsilon^{R+M-2} \varepsilon^{\alpha_1} \partial^\alpha (q_\varepsilon^R \partial_\theta v_\varepsilon^{1R}), \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle| &\leq 2C^2 \left(\sum_{0 \leq \beta \leq \alpha, 0 \leq |\beta| \leq 1} C_{\alpha,\beta} \right)^2 \|v_\varepsilon^{1R}\|_{H^{m+4}}^2 \\
&\quad + \frac{1}{2} \left(C^2 \left(\sum_{0 \leq \beta \leq \alpha, 2 \leq |\beta| \leq |\alpha|} C_{\alpha,\beta} \right)^2 \|v_\varepsilon^R\|_{H^{m+4}}^2 + 1 \right) \|q_\varepsilon^R\|_{H_{(1,\varepsilon)}^{|\alpha|}}^2.
\end{aligned}$$

◦ *Contribution of the fifth term.* The computation of the fifth term can be dealt copying the above proof. We do not give any details since it is the same. Finally joining all the estimate together we deduce the lemma. \blacksquare

•**Proof of Lemma III.31**• We have just estimated all the contributions. We can put it together to prove Lemma III.31. Let J an integer in $\{0, \dots, m+3\}$. We distinguish the proof between the particular case $J = 0$ and the case $J \in \{1, \dots, m+3\}$.

◦ *The case $J = 0$.* Multiplying Equation (III.147) by q_ε^R and integrating with respect to the space variables (θ, y) we get:

$$\begin{aligned} \frac{1}{2} \partial_t \|q_\varepsilon^R\|_{L^2}^2 &= -\langle \mathcal{H} q_\varepsilon^R, q_\varepsilon^R \rangle - \langle \mathcal{B} q_\varepsilon^R, q_\varepsilon^R \rangle - \langle \mathcal{D} q_\varepsilon^R, q_\varepsilon^R \rangle - \langle \mathcal{F} v_\varepsilon^R, q_\varepsilon^R \rangle - \langle S_\varepsilon^{0,R,N}, q_\varepsilon^R \rangle \\ &= -\langle \mathcal{B} q_\varepsilon^R, q_\varepsilon^R \rangle - \langle \mathcal{D} q_\varepsilon^R, q_\varepsilon^R \rangle - \langle \mathcal{F} v_\varepsilon^R, q_\varepsilon^R \rangle - \langle S_\varepsilon^{0,R,N}, q_\varepsilon^R \rangle. \end{aligned} \quad (\text{III.164})$$

For the particular case $J = 0$ the contribution of \mathcal{H} vanishes. For the other contribution we apply Lemmas III.33, III.34, III.35 and III.36. There exist C_B^1 , C_D^1 , C_F^1 , S_F , C_S^1 and S_S positive constants such that for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\begin{aligned} |\langle \mathcal{B} q_\varepsilon^R, q_\varepsilon^R \rangle| (t) &\leq C_B^1 \|q_\varepsilon^R(t, \cdot)\|_{L^2(\mathbb{T} \times \mathbb{R})}^2, \quad |\langle \mathcal{D} q_\varepsilon^R, q_\varepsilon^R \rangle| (t) \leq C_D^1 \|q_\varepsilon^R(t, \cdot)\|_{L^2(\mathbb{T} \times \mathbb{R})}^2, \\ |\langle \mathcal{F} v_\varepsilon^R, q_\varepsilon^R \rangle| (t) &\leq C_F^1 \|q_\varepsilon^R(t, \cdot)\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 + S_F \|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}}^2, \\ |\langle S_\varepsilon^{0,R,N}, q_\varepsilon^R \rangle| (t) &\leq C_S^1 \left(1 + \|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}}^2\right) \|q_\varepsilon^R(t, \cdot)\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 \\ &\quad + S_F \left(\|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}}^2 + \varepsilon^{2(N-R)}\right). \end{aligned}$$

Injecting those inequalities in Equation (III.164) we get for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$,

$$\begin{aligned} \frac{1}{2} \partial_t \|q_\varepsilon^R(t, \cdot)\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 &\leq C_0^1 \left(1 + \|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}(\mathbb{T} \times \mathbb{R})}^2\right) \|q_\varepsilon^R(t, \cdot)\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 \\ &\quad + C_0^3 \left(\|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}(\mathbb{T} \times \mathbb{R})}^2 + \varepsilon^{2(N-R)}\right), \end{aligned}$$

with $C_0^1 := C_B^1 + C_D^1 + C_F^1 + C_S^1$ whereas $C_0^3 := S_F + S_S$. \square

◦ *We now deal with the case $J \in \{1, \dots, m+3\}$.* We start again with Equation (III.148):

$$\begin{aligned} \frac{1}{2} \partial_t \|\varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R\|_{L^2}^2 &= -\langle \varepsilon^{\alpha_1} \partial^\alpha \mathcal{H} q_\varepsilon^R, \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle - \langle \varepsilon^{\alpha_1} \partial^\alpha \mathcal{B} q_\varepsilon^R, \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle \\ &\quad - \langle \varepsilon^{\alpha_1} \partial^\alpha \mathcal{D} v_\varepsilon^R, \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle + \langle \varepsilon^{\alpha_1} \partial^\alpha S_\varepsilon^{0,R,N}, \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle - \langle \varepsilon^{\alpha_1} \partial^\alpha \mathcal{F} v_\varepsilon^R, \varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R \rangle. \end{aligned}$$

The contribution of \mathcal{H} is no longer vanishing. We apply Lemmas III.32, III.33, III.34, III.35 and III.36. There exist $C_{\mathcal{H}}^1$, $C_{\mathcal{H}}^2$ and C_B^1 , C_B^2 , and C_D^1 , C_D^2 and C_F^1 , S_F and C_S^1 ,

C_S^2, S_S positive constants such that for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$,

$$\begin{aligned}
|\langle \mathcal{H} q_\varepsilon^R, q_\varepsilon^R \rangle| (t) &\leq C_{\mathcal{H}}^1 \|q_\varepsilon^R(t, \cdot)\|_{\dot{H}_{(1,\varepsilon)}^J(\mathbb{T} \times \mathbb{R})}^2 + C_{\mathcal{H}}^2 \|q_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^{J-1}(\mathbb{T} \times \mathbb{R})}^2, \\
|\langle \mathcal{B} q_\varepsilon^R, q_\varepsilon^R \rangle| (t) &\leq C_{\mathcal{B}}^1 \|q_\varepsilon^R(t, \cdot)\|_{\dot{H}_{(1,\varepsilon)}^J(\mathbb{T} \times \mathbb{R})}^2 + C_{\mathcal{B}}^2 \|q_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^{J-1}(\mathbb{T} \times \mathbb{R})}^2, \\
|\langle \mathcal{D} q_\varepsilon^R, q_\varepsilon^R \rangle| (t) &\leq C_{\mathcal{D}}^1 \|q_\varepsilon^R(t, \cdot)\|_{\dot{H}_{(1,\varepsilon)}^J(\mathbb{T} \times \mathbb{R})}^2 + C_{\mathcal{D}}^2 \|q_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^{J-1}(\mathbb{T} \times \mathbb{R})}^2, \\
|\langle \mathcal{F} v_\varepsilon^R, q_\varepsilon^R \rangle| (t) &\leq C_{\mathcal{F}}^1 \|q_\varepsilon^R(t, \cdot)\|_{\dot{H}_{(1,\varepsilon)}^J(\mathbb{T} \times \mathbb{R})}^2 + S_{\mathcal{F}} \|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}}^2, \\
|\langle S_\varepsilon^{0,R,N}, q_\varepsilon^R \rangle| (t) &\leq C_S^1 \left(1 + \|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}}^2\right) \|q_\varepsilon^R(t, \cdot)\|_{\dot{H}_{(1,\varepsilon)}^J(\mathbb{T} \times \mathbb{R})}^2 \\
&\quad + S_F \left(\|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}}^2 + \varepsilon^{2(N-R)}\right) \\
&\quad + C_S^2 \left(1 + \|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}}^2\right) \|q_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^{J-1}(\mathbb{T} \times \mathbb{R})}^2.
\end{aligned}$$

Plugging those estimates into Equation (III.148), we obtain:

$$\begin{aligned}
&\frac{1}{2} \partial_t \|\varepsilon^{\alpha_1} \partial^\alpha q_\varepsilon^R(t, \cdot)\|_{L^2(\mathbb{T} \times \mathbb{R})}^2 \\
&\leq \left(C_S^1 \left(1 + \|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}(\mathbb{T} \times \mathbb{R})}^2\right) + C_{\mathcal{H}}^1 + C_{\mathcal{B}}^1 + C_{\mathcal{D}}^1 + C_{\mathcal{F}}^1\right) \|q_\varepsilon^R(t, \cdot)\|_{\dot{H}_{(1,\varepsilon)}^J(\mathbb{T} \times \mathbb{R})}^2 \\
&\quad + \left(C_{\mathcal{H}}^2 + C_{\mathcal{A}}^2 + C_{\mathcal{B}}^2 + C_S^2 \left(1 + \|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}}^2\right)\right) \|q_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^{J-1}(\mathbb{T} \times \mathbb{R})}^2 \\
&\quad + (S_S + S_{\mathcal{F}}) \|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}(\mathbb{T} \times \mathbb{R})}^2 + S_S \varepsilon^{2(N-R)}.
\end{aligned}$$

On the left hand side to make the anisotropic homogeneous norm appear, we sum over all multi-index α of length J :

$$\begin{aligned}
\frac{1}{2} \partial_t \|q_\varepsilon^R(t, \cdot)\|_{\dot{H}_{(1,\varepsilon)}^J(\mathbb{T} \times \mathbb{R})}^2 &\leq C_J^1 \left(1 + \|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}(\mathbb{T} \times \mathbb{R})}^2\right) \|q_\varepsilon^R(t, \cdot)\|_{\dot{H}_{(1,\varepsilon)}^J(\mathbb{T} \times \mathbb{R})}^2 \\
&\quad + C_J^2 \left(1 + \|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}}^2\right) \|q_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^{J-1}(\mathbb{T} \times \mathbb{R})}^2 \\
&\quad + C_J^3 \left(\|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}(\mathbb{T} \times \mathbb{R})}^2 + \varepsilon^{2(N-R)}\right),
\end{aligned}$$

where constants C_J^1, C_J^2 and C_J^3 are defined as follows $C_J^1 := (J+1)(C_S^1 + C_{\mathcal{B}}^1 + C_{\mathcal{D}}^1 + C_{\mathcal{F}}^1 + C_{\mathcal{H}}^1)$, $C_J^2 := (J+1)(C_{\mathcal{H}}^2 + C_{\mathcal{A}}^2 + C_{\mathcal{B}}^2 + C_S^2)$ and $C_J^3 := (J+1)(S_{\mathcal{F}} + S_S)$. \blacksquare

3.2.2 Control over the pressure in anisotropic Sobolev norms

In this subsection we prove Proposition III.20. We prove, by induction on the size J that property $\mathcal{Q}(J)$ defined page 122 is satisfied for $J \in \{0, \dots, m+3\}$. One aspect of the proof is to give meaning to the *a priori* Inequalities (III.145) and (III.146). The usual method is to integrate those inequalities with respect to the time t and use a Gronwall lemma.

◦ *Property $\mathcal{Q}(0)$ is true.* First of all we apply Lemma III.31 for $J = 0$, there exist two positive constants C_0^1 and C_0^3 such that for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$,

$$\|q_\varepsilon^R(t, \cdot)\|_{L^2}^2 \leq \int_0^t \psi_\varepsilon^0(s) \|q_\varepsilon^R(s, \cdot)\|_{L^2}^2 ds + \varphi_\varepsilon^0(t), \quad (\text{III.165})$$

with $\psi_\varepsilon^0(t) := 2C_0^1 \left(1 + \|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}}^2\right)$, $\varphi_\varepsilon^0(t) := \int_0^t C_0^3 \left(\|v_\varepsilon^R(s, \cdot)\|_{H^{m+4}}^2 + \varepsilon^{2(N-R)}\right) ds$.

According to Proposition III.21, there exist ε_d and C_m two positive constants such that:

$$\forall \varepsilon \in]0, \varepsilon_d], \quad \forall t \in [0, T_\varepsilon^*], \quad \int_0^t \|v_\varepsilon^R(s, \cdot)\|_{H^{m+4}}^2 ds \leq C_m \varepsilon^{2w_m} t. \quad (\text{III.166})$$

So for all $\varepsilon \in]0, \varepsilon_d]$, the function φ_ε^0 is in $C^0([0, T_\varepsilon^*])$ whereas ψ_ε^0 is in $L^1([0, T_\varepsilon^*])$. To be more precise, the family $\{\varepsilon^{-2w_m} \varphi_\varepsilon^0\}_{\varepsilon \in]0, \varepsilon_d]}$ is bounded in C^0 and satisfies for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\varphi_\varepsilon^0(t) \leq C_0^3 C_m \varepsilon^{2w_m} t + C_0^3 \varepsilon^{2(N-R)} t \leq C_0^3 (C_m + 1) \varepsilon^{2w_m} t := K_0^1 \varepsilon^{2w_m} t,$$

whereas the family $\{\psi_\varepsilon^0\}_\varepsilon$ is bounded in L^1 and it satisfies:

$$\forall \varepsilon \in]0, \varepsilon_d], \quad \forall t \in [0, T_\varepsilon^*], \quad \int_0^t \psi_\varepsilon^0(s) ds \leq 2C_0^1 (1 + C_m) t \leq 2C_0^1 (1 + C_m^1) := K_0^2.$$

Thus, we can apply the Gronwall lemma for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\|q_\varepsilon^R(t, \cdot)\|_{L^2}^2 \leq \sup_{s \in [0, t]} (\varphi_\varepsilon^0(s)) \exp \left(\int_0^t \psi_\varepsilon^0(s) ds \right) \leq K_0^1 t \varepsilon^{2w_s} e^{K_0^2}.$$

□

◦ *We assume $\mathcal{Q}(J)$ is true, for some $J \in \{0, \dots, m+2\}$.* Apply Lemma III.31, there exist three positive constants C_{J+1}^1 , C_{J+1}^2 and C_{J+1}^3 such that for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\begin{aligned} \frac{1}{2} \partial_t \|q_\varepsilon^R(t, \cdot)\|_{H_{(1, \varepsilon)}^{J+1}(\mathbb{T} \times \mathbb{R})}^2 &\leq C_{J+1}^1 \left(1 + \|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}(\mathbb{T} \times \mathbb{R})}^2\right) \|q_\varepsilon^R(t, \cdot)\|_{H_{(1, \varepsilon)}^{J+1}(\mathbb{T} \times \mathbb{R})}^2 \\ &\quad + C_{J+1}^2 \left(1 + \|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}}^2\right) \|q_\varepsilon^R(t, \cdot)\|_{H_{(1, \varepsilon)}^J(\mathbb{T} \times \mathbb{R})}^2 \\ &\quad + C_{J+1}^3 \left(\|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}(\mathbb{T} \times \mathbb{R})}^2 + \varepsilon^{2(N-R)}\right). \end{aligned}$$

We integrate it with respect to the time t to write,

$$\|q_\varepsilon^R(t, \cdot)\|_{H_{(1, \varepsilon)}^{J+1}}^2 \leq \int_0^t \psi_\varepsilon^{J+1}(s) \|q_\varepsilon^R(s, \cdot)\|_{H_{(1, \varepsilon)}^{J+1}}^2 ds + \varphi_\varepsilon^{J+1}(t),$$

with for all $\varepsilon \in]0, 1]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\begin{aligned}\psi_\varepsilon^{J+1}(t) &:= C_{J+1}^1 \left(1 + \|v_\varepsilon^R(t, \cdot)\|_{H^{m+4}}^2 \right), \\ \varphi_\varepsilon^{J+1}(t) &:= \int_0^t \left(C_{J+1}^2 \left(1 + \|v_\varepsilon^R(s, \cdot)\|_{H^{m+4}}^2 \right) \|q_\varepsilon^R(s, \cdot)\|_{H_{(1,\varepsilon)}^J}^2 \right. \\ &\quad \left. + C_{J+1}^3 \left(\|v_\varepsilon^R(s, \cdot)\|_{H^{m+4}}^2 + \varepsilon^{2(N-R)} \right) \right) ds.\end{aligned}$$

First by the assumption of induction, there exist ε_d and M_J such that:

$$\forall \varepsilon \in]0, \varepsilon_d], \quad \forall t \in [0, T_\varepsilon^*], \quad \|q_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^J}^2 \leq \varepsilon^{2w_m} M_J t.$$

From Inequality (III.166) and assumption of induction, we deduce that for all $\varepsilon \in]0, \varepsilon_d]$, the function $\varphi_\varepsilon^{J+1}$ is continuous on the interval $[0, T_\varepsilon^*]$ and ψ_ε^{J+1} lies in $L^1([0, T_\varepsilon^*])$. Furthermore on the one hand the family $\{\varepsilon^{-2w_m} \varphi_\varepsilon^{J+1}\}_\varepsilon$ is bounded in C^0 , for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\begin{aligned}\varphi_\varepsilon^{J+1}(t) &= \int_0^t \left(C_{J+1}^2 \left(1 + \|v_\varepsilon^R(s, \cdot)\|_{H^{m+4}}^2 \right) \|q_\varepsilon^R(s, \cdot)\|_{H_{(1,\varepsilon)}^J}^2 \right. \\ &\quad \left. + C_{J+1}^3 \left(\|v_\varepsilon^R(s, \cdot)\|_{H^{m+4}}^2 + \varepsilon^{2(N-R)} \right) \right) ds, \\ &\leq C_{J+1}^2 \int_0^t \varepsilon^{2w_m} M_J \left(1 + \|v_\varepsilon^R(s, \cdot)\|_{H^{m+4}}^2 \right) s ds \\ &\quad + C_{J+1}^3 \int_0^t \|v_\varepsilon^R(s, \cdot)\|_{H^{m+4}}^2 ds + C_{J+1}^3 \int_0^t \varepsilon^{2(N-R)} ds, \\ &\leq (C_{J+1}^2 M_J (1 + C_m) + C_{J+1}^3 (C_m + 1)) \varepsilon^{2w_m} t := K_{J+1}^1 \varepsilon^{2w_m} t.\end{aligned}$$

On the other hand the family $\{\psi_\varepsilon^{J+1}\}_\varepsilon$ is bounded in L^1 . It satisfies:

$$\begin{aligned}\forall \varepsilon \in]0, \varepsilon_d], \quad \forall t \in [0, T_\varepsilon^*], \quad \int_0^t \psi_\varepsilon^{J+1}(s) ds &= C_{J+1}^1 \int_0^t \left(1 + \|v_\varepsilon^R(s, \cdot)\|_{H^{m+4}}^2 \right) ds, \\ &\leq C_{J+1} (1 + C_m) := K_{J+1}^2.\end{aligned}$$

Finally apply the Gronwall lemma, for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\|q_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^{J+1}}^2 \leq \sup_{s \in [0, t]} (\varphi_\varepsilon^{J+1}(s)) \exp \left(\int_0^t \psi_\varepsilon^{J+1}(s) ds \right) \leq K_{J+1}^1 t \varepsilon^{2w_m} e^{K_{J+1}^2}.$$

It proves the induction. ■

Of course an immediate corollary is Proposition III.20. We now use the \sim notation. From Subsection 3.1 and 3.2 we deduce that there exist ε_d a positive constant, C_m^1 , C_m^2 and C_m positive constants such that, for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\|\tilde{q}_\varepsilon^R(t, \cdot)\|_{H_{(1,\varepsilon)}^{m+3}(\mathbb{T} \times \mathbb{R})}^2 \leq \varepsilon^{2w_m} C_m t, \quad \|\tilde{v}_\varepsilon^R(t, \cdot)\|_{H^{m+3}(\mathbb{T} \times \mathbb{R})}^2 \leq \varepsilon^{2w_m} C_1^m \left(e^{C_m^2 t} - 1 \right).$$

Then since we have $(e^{C_m^2 t} - 1) \leq C_m^2 e^{C_m^2 t} t$ for all time $t \in [0, 1]$, it ends the proof.

3.3 Bounded life span - Comments

In this subsection we end the proof of Proposition [III.4](#). In particular we prove that there exists a positive time T_c smaller than T_ε^* for any ε (small enough). Then we comment the precision condition on the constant w_m (see Equation [\(III.13\)](#)). Here, we use the \sim notation to make things clear.

• **Bounded Life-span** • We start to prove the existence of a bounded life span for the variable $(\tilde{q}_\varepsilon^R, \tilde{v}_\varepsilon^R)$. Then, we easily go back to the original variable $(q_\varepsilon^R, v_\varepsilon^R)$ with the adequate change of variables.

Lemma III.37. *Consider integers m, ν, M, N and R satisfying $M \geq 7/2$ and the condition [\(III.13\)](#). Then, there exist two positive constants ε_c and T_c (independent of ε), and a positive constant $c_{err} > 0$ such that,*

$$\forall t \in [0, T_c], \quad \forall \varepsilon \in]0, \varepsilon_c], \quad \|\tilde{q}_\varepsilon^R(t, \cdot)\|_{H_{(1, \varepsilon)}^{m+3}(\mathbb{T} \times \mathbb{R})} \leq c_{err}, \quad \|\tilde{v}_\varepsilon^R(t, \cdot)\|_{H^{m+3}(\mathbb{T} \times \mathbb{R})} \leq c_{err}.$$

Proof of Lemma III.37. We argue by contradiction:

$$\forall (\tilde{\varepsilon}, T) \in]0, 1] \times [0, 1], \quad \exists \varepsilon \in]0, \tilde{\varepsilon}], \quad T_\varepsilon^* < T.$$

◦ We recall that there exist ε_d and C_m positive constants such that, for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\|\tilde{q}_\varepsilon^R(t, \cdot)\|_{H_{(1, \varepsilon)}^{m+3}(\mathbb{T} \times \mathbb{R})}^2 \leq \varepsilon^{2w_m} C_m t, \quad \|\tilde{v}_\varepsilon^R(t, \cdot)\|_{H^{m+3}(\mathbb{T} \times \mathbb{R})}^2 \leq \varepsilon^{2w_m} C_m t.$$

We choose for instance $T = \min(\frac{1}{2C_m}, \frac{1}{2})$ and $\tilde{\varepsilon} = \varepsilon_d < 1$. In particular, $T < 1$. From assumption, there exists $\varepsilon_0 \in]0, \varepsilon_d]$ such that $T_{\varepsilon_0}^* < T$. Furthermore, since $w_m \geq 0$ from condition [\(III.13\)](#) we get:

$$\forall t \in [0, T_{\varepsilon_0}^*], \quad \|\tilde{q}_{\varepsilon_0}^R(t, \cdot)\|_{H_{(1, \varepsilon_0)}^{m+3}(\mathbb{T} \times \mathbb{R})} \leq \frac{1}{2} < 2, \quad \|\tilde{v}_{\varepsilon_0}^R(t, \cdot)\|_{H^{m+3}(\mathbb{T} \times \mathbb{R})} \leq \frac{1}{2} < 2.$$

◦ Now consider the applications

$$t \in [0, T_{\varepsilon_0}^*] \mapsto \|\tilde{q}_{\varepsilon_0}^R(t, \cdot)\|_{H_{(1, \varepsilon_0)}^{m+3}(\mathbb{T} \times \mathbb{R})} \quad \text{and} \quad t \in [0, T_{\varepsilon_0}^*] \mapsto \|\tilde{v}_{\varepsilon_0}^R(t, \cdot)\|_{H^{m+3}(\mathbb{T} \times \mathbb{R})}.$$

They are continuous. $\tilde{q}_{\varepsilon_0}^R$ (respectively $\tilde{v}_{\varepsilon_0}^R$) can be extended in time as long as the quantity $\|\tilde{q}_{\varepsilon_0}^R(t, \cdot)\|_{H_{(1, \varepsilon_0)}^{m+3}}$ (respectively $\|\tilde{v}_{\varepsilon_0}^R(t, \cdot)\|_{H^{m+3}}$) remains bounded.

It follows that we can find $T \in]T_{\varepsilon_0}^*, T_{\varepsilon_0}]$ such that for all time $t \in [0, T]$, $\|\tilde{q}_{\varepsilon_0}^R(t, \cdot)\|_{H_{(1, \varepsilon_0)}^{m+3}} < 2$ (respectively $\|\tilde{v}_{\varepsilon_0}^R(t, \cdot)\|_{H^{m+3}} < 2$). This is in contradiction with the definition of $T_{\varepsilon_0}^*$. ■

• **Conclusion** • Returning to the initial unknown $\tilde{q}_\varepsilon^R = q_\varepsilon^R$, $\tilde{v}_\varepsilon^R = v_\varepsilon^R$ and $\tilde{v}_\varepsilon^{2R} = \varepsilon v_\varepsilon^{2R}$, we obtain for all time $t \in [0, T_c]$ and for all $\varepsilon \in]0, \varepsilon_c]$,

$$\|q_\varepsilon^R(t, \cdot)\|_{H_{(1, \varepsilon)}^{m+3}(\mathbb{T} \times \mathbb{R})} \leq c_{err}, \quad \|v_\varepsilon^{1R}(t, \cdot)\|_{H^{m+3}(\mathbb{T} \times \mathbb{R})} \leq c_{err}, \quad \|\varepsilon v_\varepsilon^{2R}(t, \cdot)\|_{H^{m+3}(\mathbb{T} \times \mathbb{R})} \leq c_{err}. \quad (\text{III.167})$$

In other words, we prove the existence of two positive constants ε_c and T_c such that:

$$\forall \varepsilon \in]0, \varepsilon_c], \quad T_\varepsilon \geq T_c.$$

Moreover from the assumption (III.167), we obtain the announced bounds:

$$\begin{aligned} \sup_{\varepsilon \in]0, \varepsilon_c]} \sup_{t \in [0, T_c]} \|q_\varepsilon^R(t, \cdot)\|_{H_{(1, \varepsilon)}^{m+3}} &< +\infty, \\ \sup_{\varepsilon \in]0, \varepsilon_c]} \sup_{t \in [0, T_c]} \|v_\varepsilon^{1R}(t, \cdot)\|_{H^{m+3}} &< +\infty, \quad \sup_{\varepsilon \in]0, \varepsilon_c]} \sup_{t \in [0, T_c]} \|\varepsilon v_\varepsilon^{2R}(t, \cdot)\|_{H^{m+3}} < +\infty. \end{aligned}$$

We prove of Proposition III.4.

•**Remarks on conditions $M \geq 7/2$ and (III.13)**• Most of the remarks on those two conditions have ever been made. We summarize them here.

◦ The first condition $M \geq 7/2$ is a technical assumption. We have seen in Subsection 3.2 that we need to control derivatives of the pressure such as $\partial_\theta q_\varepsilon^R$. This derivative is not bounded in the anisotropic Sobolev spaces $H_{(1, \varepsilon)}^m$. We overcame this difficulty passing a power of ε from ε^{M-2} to terms such as $\partial_\theta q_\varepsilon^R$. Thus, it requires $M \geq 3$.

However, the lost of $\sqrt{\varepsilon}$ seems technical. It appears when we use the embeddings $W_{(1, \varepsilon)}^{1, \infty} \hookrightarrow H_{(1, \varepsilon)}^m$. The result should still be true for $M \geq 3$.

◦ The condition on the parameter w_m can be divided in two:

- The first condition is $R \leq N$. It just means that the approximated solution can not be more accurate the approximation of the equation $\mathcal{L} = 0$ obtained.
- The second condition $2\nu - M - 5/2 - (m + 3) - R \geq$ is more delicate to discuss. First of all we can notice that it requires less precision than the condition (III.12). Indeed it requires $m + 3$ derivatives for w_m instead of $m + 4$ for (III.12). It is due to the regularization phenomena of the viscosity. Indeed we passed one derivative of $q \nabla q$ onto the velocity by integration by parts.

The loss of precision $-(m + 3)$ is due to the choice of different norms for the pressure (anisotropic norms) and the viscosity (classical Sobolev norms). It implies some difficulty when dealing with interactions of the pressure and the velocity. The interactions are traduced in Equation (III.111b) through the term $q \nabla q$. We still control it thanks to the equivalence of norms (III.106) to a cost of order $m + 3$.

Choosing the anisotropic spaces $H_{(1, \varepsilon)}^m$ for both variables, this condition is transformed into

$$2\nu - M - 5/2 \geq R.$$

If ν is large enough we obtain a control up to integer R . For instance if $N = R$ this inequality becomes:

$$2\nu - M - 5/2 \geq N.$$

Thus, it justifies our approach to obtain an approximated solution, if the nonlinear term $\varepsilon^{2\nu} q \nabla q$ is weak enough with respect to ε^N (*i.e.* ν is large with respect to N). Since we construct the approximated solution not taking into account $\varepsilon^{2\nu} q \nabla q$ we can't hope to obtain a good approximation if N is too large and interfere with this nonlinear term.

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Chapter IV

Perspectives

1 Sur l'équation de Helmholtz haute-fréquence

1.1. Afin d'éviter de traiter les difficultés induites par le bord de la variété stationnaire M (définie page 27), on a recours à l'hypothèse technique :

$$\text{supp}(\widehat{S}) \subset \{t.\eta \text{ s.t. } \eta \in I_{\theta_0}, t > 0\}. \quad (\text{IV.1})$$

Une extension naturelle serait d'éliminer cette contrainte de radiation (IV.1). On aurait alors une convergence au sens des distributions tempérées de w_ε vers w_{out} pour toute source.

1.2. Dans [7], on suppose deux conditions pour montrer le résultat de convergence : l'hypothèse de *non-refocalisation des rayons* (II.10) et l'hypothèse de *transversalité des rayons* (II.18). Nous avons uniquement discuté dans cette thèse de l'hypothèse de *non-refocalisation des rayons*. L'optimalité de l'hypothèse de *transversalité des rayons* reste à démontrer.

1.3. Une extension du résultat [7] serait de rajouter une dépendance en la variable d'espace x pour le coefficient d'absorption α_ε voire d'autoriser une dégénérescence (annulation de la fonction) sur des sous-ensembles de \mathbb{R}^d . Son rôle dans l'équation est très important puisqu'il sélectionne la solution sortante de l'équation d'Helmholtz. Si α_ε reste positif, on s'attend encore à ce que w_ε (c.f. (I.2)) converge toujours vers w_{out} solution de l'équation (I.4). Mais cela n'a pas été démontré.

2 Sur l'équation de Navier-Stokes forcée

2.1. La condition technique $M \geq 7/2$ requise au niveau de la Proposition (III.4) pour le résultat de stabilité s'améliore sans doute jusqu'à $M \geq 3$. Par contre, le passage de $M \geq 3$ à $M \geq 2$ devrait être moins évident. Ce décalage fait que la gestion de la partie hyperbolique singulière des équations (en particulier la gestion des normes anisotropes) tout en exploitant les informations apportées par la dissipation semble remise en cause.

2.2. Les oscillations forcées dans la direction x_1 imposent une étude du problème sur $\mathbb{T} \times \mathbb{R}$ (en espace). D'autres modèles (peut-être Shallow-water) pourraient amener à une étude du

problème sur un domaine borné Ω en espace. En ce cas, l'existence d'une couche limite en temps exponentiellement décroissante (liée au trou spectral dans le spectre de $-\mu\partial_{\theta\theta}$) peut alors être remise en cause. Dans ce contexte, la construction d'une solution approchée devrait nécessiter d'autres arguments.

2.3. Les interactions entre la pression et la viscosité ont été négligées en supposant une très faible pression (ν très grand). Ces interactions pourraient être renforcées par une diminution du paramètre ν . La décomposition en deux couches limites pour la vitesse serait alors beaucoup moins évidente. Par ailleurs, il semblerait que l'échelle de temps t/ε introduite par la pression joue un rôle dans ces interactions.

2.4. Les mécanismes d'interactions d'ondes ont été mis en avant pour un problème de stabilité près d'une solution exacte qui ne dépend que de la variable d'espace θ . Le problème de stabilité pourrait être étudié pour une solution stationnaire de l'équation de Navier-Stokes forcée, voire près d'une solution exacte.

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MÉTHODES ASYMPTOTIQUES POUR LES ÉQUATIONS DE TYPE HELMHOLTZ OU NAVIER-STOKES

Résumé

Dans cette thèse, nous étudions deux problèmes différentiels dépendant d'un paramètre ε et étudions l'asymptotique des solutions lorsque ce paramètre tend vers 0.

Le premier problème est lié à l'équation de Helmholtz haute-fréquence. On construit un potentiel non captif ne satisfaisant pas l'hypothèse de refocalisation des rayons introduite par F. Castella. On montre que l'ensemble des trajectoires hamiltoniennes (associées au potentiel construit) issues de l'origine et qui reviennent en 0 forme une sous-variété de dimension $d - 1$, où d est la dimension de l'espace.

On montre alors que la solution de l'équation de Helmholtz converge vers une perturbation de la solution de Helmholtz avec condition de radiation à l'infini et coefficients figés en 0.

Dans un second temps, nous étudions une équation de Navier-Stokes forcée par une source polarisée fortement oscillante. On exhibe une famille de solutions exactes. On étudie alors la stabilité de cette famille lorsqu'on la perturbe à l'instant initial. On construit une solution approchée du problème à l'aide d'une couche limite à l'instant initial ($t=0$). Ce développement montre en particulier que des interactions d'ondes, se propageant à des échelles différentes, peuvent se traduire au niveau macroscopique par une augmentation de la viscosité.

Enfin, on justifie la convergence de la solution approchée vers la solution exacte à l'aide de méthodes d'énergie.

ASYMPTOTICAL METHODS FOR HELMHOLTZ OR NAVIER-STOKES TYPE EQUATIONS

Abstract

In this thesis, we study two differential problems which depend on a small parameter ε . We study the asymptotic of the solutions when ε tends to 0.

The first problem deals with the high-frequency Helmholtz equation. We construct a non-trapping potential which does not satisfy the refocusing condition introduced by F. Castella. We prove that the Hamiltonian trajectories (associated with this potential) issued from 0 which go back to the origin form a submanifold of dimension $d - 1$, where d denotes the space dimension.

We show that the solution converges to a perturbation of the out-going solution with coefficients frozen at 0.

Then we study a Navier-Stokes type equation forced by a polarised and oscillating source. We exhibit a family of exact solutions to the problem. We study the stability of this solution when we perturb it at the initial time. We construct an approximated solution of this problem thanks to a boundary layer in time in $t = 0$. In particular, it shows that interactions of oscillating waves, which propagate at different scales, can be modelised at macroscopic scales by some creation of dissipation.

Finally, we justify the convergence of the approximated solution towards the exact solution by performing some energy methods.